Lecture 26

<u>Thom Isomorphism</u> ($\mathbb{Z}_2$) Given $F \hookrightarrow E \xrightarrow{} B$ (i.e. $E \downarrow B$), the group $H^i(E, E_0; \mathbb{Z}_2)$ is zero for $i<n$, and $H^n(E,E_0;\mathbb{Z}_2)$ contains a unique class u st $\forall\ F = p^{-1}(b)$, the restriction $u|_{(F,F_0)} \in H^n(F,F_0;\mathbb{Z}_2)$ is the unique nonzero class in $H^n(F,F_0;\mathbb{Z}_2)$. Moreover

$$H^k(E;\mathbb{Z}_2) \xrightarrow{\sim} H^{k+n}(E,E_0;\mathbb{Z}_2)$$
$$x \longmapsto x\cup u \qquad \forall k.$$

<u>Proof</u>

<u>Step one</u> There is an isomorphism

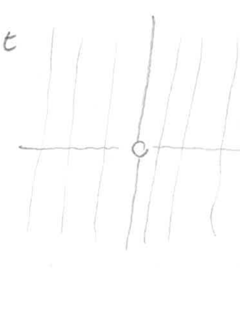
$$\tilde H^j(B) \longrightarrow \tilde H^{j+1}(B\times\mathbb{R}, B\times\mathbb{R}_0),$$
$$x \longrightarrow x\times e^1$$

where e^1 is the unique nonzero element in $\hat H^j(\mathbb{R},\mathbb{R}_0)$. (Les of the triple $(B\times\mathbb{R}, B\times\mathbb{R}_0, B\times\mathbb{R}_-)$.

<u>Step Two</u> There is a chain of isomorphisms

$$\begin{array}{ccccccc} \tilde H^j(B) & \to & H^{j+1}(B\times\mathbb{R}, B\times\mathbb{R}_0) & \to & H^{j+2}(B\times\mathbb{R}^2, B\times\mathbb{R}_0^2) & \to \dots \to & H^{j+n}(B\times\mathbb{R}^n, B\times\mathbb{R}^n_0) \\ x & & x\times e^1 & & x\times e^2 & & x\times e^n \end{array}$$

via proving that for (C,C'), $H^j(C,C') \xrightarrow{\sim} H^{j+1}(C\times\mathbb{R}, (C'\times\mathbb{R})\cup(C\times\mathbb{R}_0))$ (Five lemma) and taking (eg) $C = B\times\mathbb{R}$, $C' = (B\times\mathbb{R}_0)$ at the first step.



Step Three Say $\begin{array}{c} E = B\times\mathbb{R}^n \\ \downarrow \\ B \end{array}$. Then $H^n(E, E_0) \simeq H^n(B\times\mathbb{R}^n, B\times\mathbb{R}^n_0) \simeq H^0(B)$.

Set $u = 1\times e^n$. $H^i(B\times\mathbb{R}^n) \longrightarrow H^i(B\times\mathbb{R}^n, B\times\mathbb{R}^n_0)$

$$y\times 1 \longmapsto (y\times 1)\cup u = (y\times 1)\cup(1\times e^n) = y\times e^n.$$

Step Four $B = B'\cup B''$ where the theorem is true over each set and their intersection.

M-V sequence

$$\cdots \to H^{n-1}(E'\cap E'', (E'\cap E'')_0) \to H^n(E, E_0) \to \overset{u'}{H^n(E', E'_0)} \oplus \overset{u''}{H^n(E'', E''_0)} \to H^n(E'\cap E'', (E'\cap E'')_0)$$

By uniqueness, u' and u'' map to the same element w/ $u'\oplus u'' \mapsto 0 \Rightarrow$ $\exists$ a u in the preimage. For the Thom isomorphism we now use the five lemma.

Step Five Finitely many B_i. Induction.

Step Six Infinitely many B_i. Certainly the thm is true for $E|_C$ where C is cpt.

Moreover $H^j(B) \xleftarrow{\sim} \varprojlim H^j(C;\mathbb{Z}_2)$ and $H^j(E, E_0;\mathbb{Z}_2) \xleftarrow{\sim} H^j(p^{-1}(C), p^{-1}(C)_0)$

[This depends on using a field: $\varinjlim H_j(C) \simeq H_j(B)$ is always true, and over a field we have no Tor, so taking cohomology we get $\mathrm{Hom}(\varinjlim H_j(C);\mathbb{Z}_2) \xrightarrow{\sim} \varprojlim \mathrm{Hom}(H_j(C),\mathbb{Z}_2)$.]

So $H^n(E,E_0) \xrightarrow{\sim} \varprojlim \left(H^n(p^{-1}(c), p^{-1}(c)_0\right)$

one class u here

One class here for each c s.t. restriction to the fibres is nonzero.

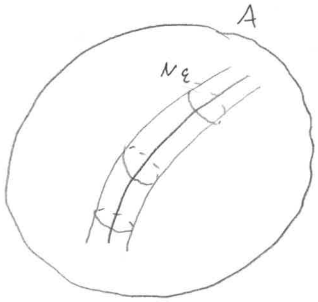
Now
$$\begin{tikzcd} H^j(E) \arrow[r, "\cup u", "\sim"'] \arrow[d, "\wr"] & H^{j+n}(E,E_0) \arrow[d, "\wr"] \\ H^j(p^{-1}(c)) \arrow[r, "\sim"] & H^{j+n}(p^{-1}(c), p^{-1}(c)_0) \end{tikzcd}$$
induces the cup product isomorphism,

Euler Class and Smooth Manifolds

The fundamental class of a smooth submfd:

Let $M^n \subseteq A^{n+k}$ closed, Riemannian.

A



Recall $\exists$ a tubular nbhd of M which is isomorphic to NM (via exponentiation).

$H^*(NM, NM_0; \Lambda) \simeq H^*(A, A-M;\Lambda)$ by excision.

So $\exists$ a fundamental class $u' \in H^k(A, A-M;\Lambda)$.

<u>Thm</u> $H^k(A, A-M;\mathbb{Z}_2) \longrightarrow H^k(A;\mathbb{Z}_2) \longrightarrow H^k(M;\mathbb{Z}_2)$

$u' \longmapsto u'|_A \longrightarrow w_k(NM)$

<u>Pf</u> Let $s: M \to E$ be the zero section of some bundle over M.

$H^k(E, E_0;\mathbb{Z}_2) \to H^k(E;\mathbb{Z}_2) \xrightarrow{s^*} H^k(M;\mathbb{Z}_2)$

$u \longmapsto u|_E \longrightarrow w_k(\xi)$

[Recall $\phi: H^k(M;\mathbb{Z}_2) \longrightarrow H^{2k}(E,E_0;\mathbb{Z}_2)$ applied to $s^*(u|_E)$ is $p^*s^*(u|_E) \cup u =$

$= u|_E \cup u = u \cup u = Sq^k(u) \Rightarrow s^*(u|_E) = q^{-1}(Sq^k(u)) = w_k(\xi).$]

So the class $u' \in H^k(A, A-M; \mathbb{Z}_2)$ corresponding to $w_k'(NM)$. (NM).

<u>Defn</u> $u'|_A \in H^k(A)$ is the <u>dual cohomology class</u>.

<u>Corollary</u> M closed, $\partial M = \phi$. If $M = M^n \hookrightarrow \mathbb{R}^{n+k}$ smoothly then $w_k(NM)$ vanishes. Ergo if $\bar{w}_k(TM) \neq 0$, $M^n \not\hookrightarrow \mathbb{R}^{n+k}$.

Say M^n smooth. Then $T(M \times M) \simeq TM \times TM$.

$$(x, y, e_1, e_2) \simeq (x, e_1) \times (y, e_2)$$

We have $\Delta : M \hookrightarrow M \times M$ w/ $TM \simeq NM$.

$$(x, x, e, e) \simeq (x, x, e, -e)$$

We want to think about the oriented case.

<u>Lemma</u> Orientation for $TM \Leftrightarrow$ Orientation for M.

$$H_n(M, M-x; \mathbb{Z}) \simeq H_n(TM, TM_0; \mathbb{Z})$$

via $x \hookrightarrow M$ w/ normal bundle $(TM)_x$.

For M oriented we have a fundamental class $u' \in H^n(M \times M, M \times M - \Delta(M))$ with the property that $u'|_{\Delta(M)} = e(NM) = e(TM)$.

How should we understand this class?

What is this class?

Each $H^n(M, M-x)$ has a preferred u_x w/ $\langle u_x, u_x \rangle = 1$. $\exists$ an embedding $j_x : (M, M-x) \longrightarrow (M\times M, M\times M - \Delta(M))$.

$$y \longmapsto (x, y)$$

Lemma The class $u' \in H^n(M\times M, M\times M - \Delta(M))$ is uniquely characterized by the property that $j_x^*(u') = u_x$ $\forall x \in M$.

Pf Let $U \subseteq TM_x$ be a small nbhd of 0. We have a map

$$\begin{aligned}
(U, U - \{0\}) &\hookrightarrow (M\times M, M\times M - \Delta(M)) \\
e &\longmapsto (\mathrm{Exp}(x, -e), \mathrm{Exp}(x, e)) \\
&\qquad \simeq \\
e &\longmapsto (x, \mathrm{Exp}(x, e))
\end{aligned}$$

This is j_x composed with $(U, U - \{0\}) \hookrightarrow (M, M-x)$.

More about the diagonal class

Let $u'' = u'|_{M\times M}$ be the diagonal cohomology class in $H^n(M\times M)$.

Lemma For any $a \in H^*(M)$, $(a\times 1) \smile u'' = (1\times a) \smile u''$.

Pf Let $N_\varepsilon \supseteq M$ be a tubular nbhd of $\Delta(M)$.

$$\begin{tikzcd}
& M\times M \ni (x,y) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
M & & M
\end{tikzcd}$$

p_1, p_2 agree on $M \Rightarrow$ htpc on N_ε a vector bundle over M. Ergo $p_1^*(a) = a\times 1$ and $p_2^*(a) = 1\times a$

Look at

$$\begin{tikzcd} H^i(M\times M) \arrow[r] \arrow[d, "\cup u'"] & H^i(N_\varepsilon) \arrow[d, "\cup u'|_{(\ )}"] \\ H^{i+n}(M\times M, M\times M-\Delta(M)) \arrow[r, "\sim"] & H^{i+n}(N_\varepsilon, N_\varepsilon - \Delta(M)) \end{tikzcd}$$

$(a\times 1)\cup u' = (1\times a)\cup u'$

$\square$

Let Λ be a Field.

We have $H^{p+q}(X\times Y;\Lambda)\otimes H_q(Y;\Lambda) \longrightarrow H^p(X;\Lambda)$ the <u>slant product</u>.

How does this work? $H^*(X\times Y;\Lambda) \simeq H^*(X;\Lambda)\otimes H^*(Y;\Lambda)\otimes H_*(Y;\Lambda) \longrightarrow H^*(X;\Lambda)$

$$\underbrace{a\otimes b}_{p}\otimes B \longrightarrow a\langle b, B\rangle$$

$p\otimes B$

<u>Exercise</u> This is left-linear in the sense that $((a\times 1)\cup p)/B) = a\cup (p/B)$

<u>Lemma</u> Say M is compact. Then if $\mu\in H_n(M;\Lambda)$ is the fundamental homology class, the diagonal class $u''\in H^n(M\times M;\Lambda)$ is related to μ by $u''/\mu = 1 \in H^0(M;\Lambda)$.

<u>Pf</u> WTS u''/μ restricts to 1 under all maps $H^0(M;\Lambda)\longrightarrow H^0(x;\Lambda)$.

$$\begin{tikzcd} u'' \arrow[d, mapsto] & H^n(M\times M;\Lambda) \arrow[r, "/\mu"] \arrow[d] & H^0(M;\Lambda) \arrow[d] \\ 1\times \iota_x^*(u'') & H^n(x\times M) \arrow[r, "/\mu"'] & H^0(x;\Lambda) \end{tikzcd}$$

$\iota_x : M \hookrightarrow M\times M$

$(1\times \iota_x^*(u''))/\mu = \langle \iota_x^*(u''), \mu\rangle$

Now

$$\begin{array}{ccc} \mu \longmapsto \mu_x & & \\ M \equiv (M, M - \{x\}) & & \\ \downarrow i_x & & \downarrow j_x \\ M\times M \equiv (M\times M, M\times M - \Delta(M)) & & \end{array}$$

$$\begin{aligned} \langle i_x^*(u''), \mu \rangle &= \langle j_x^*(u')|_{M}, \mu \rangle \\ &= \langle j_x^*(u'), \mu_x \rangle \\ &= 1 \end{aligned}$$

$\Rightarrow (u''/\mu)|_x = 1 \in H^0(x; \Lambda)$. $\blacksquare$

Recall

Poincaré Duality Let M cpt, smooth. Over a field given a basis $b_1, \dots, b_r$ for $H^*(M;\Lambda)$, there is a dual basis $b_1^\#, \dots, b_r^\#$ for $H^*(M;\Lambda)$ w/ $\langle b_i \cup b_j^\#, \mu \rangle = \delta_{ij}$.

Propn With respect to this basis, the diagonal class is

$$u'' = \sum_{i=1}^{r} (-1)^{\dim b_i} b_i \times b_i^\#.$$

From this we get a relationship to Euler characteristic

Recall

$$\begin{aligned} \chi(Y) &= \sum_k (-1)^k H^k(Y) \\ &= \sum (-1)^k (\#\text{ of } k\text{-cells}) \end{aligned}$$

Corollary M smooth, cpt, oriented $\Rightarrow \langle e(TM), \mu \rangle = \chi(M)$

$$\langle w_n(TM), \mu \rangle = \chi(M) \bmod 2$$

Proof $e(TM) = \Delta^*(u'')$

Rational coefficients $\rightsquigarrow u'' = \sum (-1)^{\dim b_i} b_i \times b_i^\#$

$$e(TM) = \sum (-1)^{\dim b_i} b_i \cup b_i^\#$$

$$\langle e(TM), \mu \rangle = \sum_i (-1)^{|b_i||b_i|} = \chi(M) \quad \square$$

Proof of Propn $\quad H^*(M \times M; \Lambda) \simeq H^*(M; \Lambda) \otimes H^*(M; \Lambda)$

u''

$$u'' = b_1 \times c_1 + \cdots + b_r \times c_r \quad \text{where } \dim(b_r) + \dim(c_r) = n$$

Now $(a \times 1) \cup u'' = (1 \times a) \cup u''$

$$\big\downarrow /\mu \qquad\qquad \big\downarrow /\mu$$

$= a \cup (u''/\mu)$

$= a$

$$= \sum (-1)^{|a||b_j'|} (b_j' \times (a \cup c_j'))/\mu$$

$$= \sum (-1)^{|a||b_j'|} (b_j' \langle a \cup c_j', \mu \rangle)$$

Taking $a = b_i$, we have $b_i = \sum (-1)^{|b_i||b_j|} b_j \langle b_i \cup c_j, \mu \rangle$

If $i = j$, we have $\langle b_i \cup c_j, \mu \rangle = (-1)^{|b_i|^2}$.

If $i \neq j$, we have $\langle b_i \cup c_j, \mu \rangle = 0$.

Set $b_i^\# = (-1)^{|b_i|} c_j$.