Lecture 25

When does a bundle have a nowhere zero section?

- We know the Stiefel-Whitney classes don't detect this; a bundle with $w_n = 0$ may or may not have a section.

Let $V_k'(F)$ be the Stiefel mfd of k-frames in a fibre of $\xi \left\{ \begin{array}{c} E \\ \downarrow \\ B \end{array} \right.$, and $V_k'(\xi)$ be the corresponding fibre bundle. I can always get a section over B^0.

$$\begin{array}{ccc} V_k'(F) & \hookrightarrow & V_k'(\xi) \\ & & \downarrow \\ & & B \end{array}$$

$$\begin{array}{c} E \\ \Big\downarrow \\ B = \{B^0 \subseteq B^1 \subseteq B^2 \subseteq \dots\} \end{array}$$

Let's say I know my section over B^i. This gives me a map

$$\underbrace{S^i}_{D^{i+1}} \longrightarrow V_1'(\xi)$$

Now $V_k'(F)$ is in general $(n-k-1)$-connected, so in particular $V_1'(F)$ is $(n-2)$-connected. Since ξ is trivializable over D^{i+1}, this is the connectivity that matters. So the first place we might not be able to extend our section is at the n-cells. (where $i = n-1$).

We get a class in $H^n(B; \underbrace{\{\pi_{n-1}(V_1'(F))\}}_{\text{These are } \mathbb{Z}\text{'s, but we worry about identifying them.}})$

Remember this def retracts onto S^{n-1}.

The <u>Euler class</u> is the obstruction to extending a section over the n-skeleton of B.

<u>Construction (Eventually)</u>

$$\begin{array}{ccc} F & \longrightarrow & E \\ \cong & & \downarrow \\ \mathbb{R}^n & & B \end{array}$$

$F_0 = F - \{0\}$

$E_0 = E - (B \times \{0\})$

$\hat{H}^i(F, F_0; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & n = i \\ 0 & \text{otherwise} \end{cases}$

$\hat{H}^i(F, F_0; \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = i \\ 0 & \text{otherwise} \end{cases}$

<u>Start w/ $\mathbb{Z}_2$</u>

<u>Thm</u> (Thom Isomorphism) The group $H^i(E, E_0; \mathbb{Z}_2)$ is zero for $i < n$, and $H^n(E, E_0; \mathbb{Z}_2)$ contains a unique class a unique class u ST for each $F = p^{-1}(b)$, the restriction $u|_{(F, F_0)} \in H^n(F, F_0; \mathbb{Z}_2) \in H^n(F, F_0; \mathbb{Z}_2)$ is the unique nonzero class in $H^n(F, F_0; \mathbb{Z}_2)$. Moreover

$$H^k(E; \mathbb{Z}_2) \longrightarrow H^{k+n}(E, E_0; \mathbb{Z}_2)$$ is an isomorphism $\forall$ k.

$$x \longmapsto x \cup u$$

<u>Defn</u> The <u>Thom isomorphism</u> is

$$\phi : H^k(B; \mathbb{Z}_2) \xrightarrow{p^*} H^k(E; \mathbb{Z}_2) \longrightarrow H^{k+n}(E, E_0; \mathbb{Z}_2)$$

What is this good for?

In $\mathbb{Z}_2$: Can give another construction of w_i.

Propn $\exists$ a set of cohomology operations called the Steenrod squares w/ the properties

1. $\forall\ X \subseteq Y$ cw cpxes and $n, i \in \mathbb{Z}_{\geq 0}$, $\exists\ Sq^i : H^n(X, Y; \mathbb{Z}_2) \to H^{n+i}(X, Y; \mathbb{Z}_2)$

2. Naturality $f : (X, Y) \longrightarrow (X', Y')$ then $Sq^i \circ f^* = f^* \circ Sq^i$.

3. Dimension If $a \in H^n(X, Y)$, then $Sq^0(a) = a$, $Sq^n(a) = a \cup a$, $Sq^i(a) = 0$ for $i > n$.

4. Cartan Formula $Sq^k(a \cup b) = \sum_{i+j=k} Sq^i(a) \cup Sq^j(b)$

This gives another defn of the Stiefel-Whitney classes

$$\phi : H^i(B; \mathbb{Z}_2) \longrightarrow H^{i+n}(E, E_0; \mathbb{Z}_2)$$

$$w_i(\xi) \longmapsto Sq^i(u)$$

$w_i(\xi) = \phi^{-1} Sq^i \phi(1)$ $\qquad$ $w_n(\xi) = \phi^{-1}(u \cup u)$

HW A concrete construction of the Steenrod squares from E-M spaces.

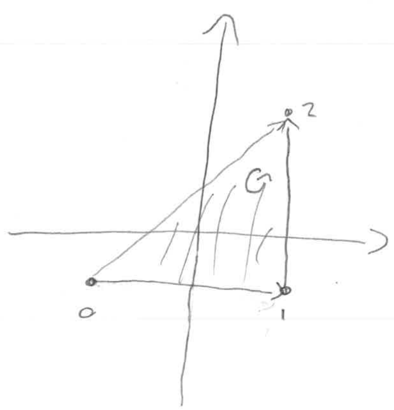
Note relation to our first construction: $(V, V_0) \simeq S^n \longrightarrow \mathbb{R}P^{n-1}$

Looking at behavior of fundamental class.

What else is this good for?

There is an analog in $\mathbb{Z}$ coefficients, when there exists a fundamental class in $H^n(E, E_0; \mathbb{Z})$.

<u>Defn</u> An <u>orientation</u> of a real vector space F of dimn $n > 0$ is an equivalence class of ordered bases; Two orientations are equivalent $\Leftrightarrow$ they are related by a matrix of positive determinant.



Corresponds to picking a generator in $H_n(F, F_0; \mathbb{Z}) \simeq \mathbb{Z}$.

An orientation for $E \xrightarrow{p} B$ is an orientation for each fibre st $\forall\ b \in B$ $\exists$ a nbhd U w/ $p^{-1}(U) \simeq U \times \mathbb{R}^n$

$$\begin{tikzcd} p^{-1}(U) \simeq U \times \mathbb{R}^n \arrow[d, bend right] \\ U \arrow[u, bend right, "s_1, \dots, s_n"'] \end{tikzcd}$$

$\Big\}$ Sections determining the orientation

So each $H^n(F, F_0; \mathbb{Z})$ has a preferred generator u_F and $\forall\ b$ $\exists\ U \ni b$ st $u \in H^n(p^{-1}(U), p^{-1}(U)_0; \mathbb{Z})$ has a preferred class st $u|_{(F, F_0)} = u_F$.

Thm Let $E \to B$ n dim'l real oriented vector bundle. Then $H^i(E, E_0; \mathbb{Z}) = 0$ for $i < n$, and $H^n(E, E_0; \mathbb{Z})$ contains a unique u st $u|_{(F, F_0)} \in H^n(F, F_0; \mathbb{Z})$ for ever fiber F of V. Moreover $H^n(E; \mathbb{Z}) \xrightarrow{\sim} H^{k+n}(E, E_0; \mathbb{Z})$.

Again we have a Thom Isomorphism $\Phi: H^k(B; \mathbb{Z}) \longrightarrow H^{k+n}(E, E_0; \mathbb{Z})$.

$$x \longmapsto p^* x \cup u$$

Now consider $(E, \phi) \hookrightarrow (E, E_0)$.

$$H^*(E, E_0; \mathbb{Z}) \to H^*(E; \mathbb{Z}) \simeq H^*(B; \mathbb{Z})$$

$$u \longrightarrow u|_E = e(\xi)$$ } This is the Euler class.

Properties

① Clearly natural for orientation-preserving

$$\begin{array}{ccc} E_1 & \longrightarrow & E_2 \\ \downarrow & & \downarrow \\ B_1 & \longrightarrow & B_2 \end{array}$$

In particular if E_1 is trivial, take E_2 to be a bundle over a point $\rightsquigarrow e(\xi_1) = 0$.

② $e(\xi) = -e(-\xi_1)$

③ If ξ has odd fibre dimension, $2e(\xi)=0$. [Often this means $e(\xi)=0$.]

Pf $(b,e)\mapsto(b,-e)$ is an orientation-reversing vector bundle isomorphism.

Lemma If $E\to B$ has a nowhere-zero section, then $e(\xi)=0$.

Pf
$$\begin{tikzcd} B \arrow[r, "s"] \arrow[rrr, bend right, "id"'] & E_0 \arrow[r, hook] & E \arrow[r, "p"] & B \end{tikzcd}$$

$$\begin{array}{ccccccc} H^n(B) & \xrightarrow{p^*} & H^n(E) & \longrightarrow & H^n(E_0) & \xrightarrow{s^*} & H^n(B) \\ e(\xi) & \longrightarrow & u|_E & \longrightarrow & (u|_E)|_{E_0} & \longrightarrow & 0 \end{array}$$

This happens b/c u is a class in
$$\begin{array}{ccccc} H^n(E,E_0) & \longrightarrow & H^n(E) & \longrightarrow & H^n(E_0) \\ u & \longrightarrow & u|_E & \longrightarrow & 0 \end{array}$$

Erg $e(\xi)=0$.

We should have some nonzero examples.

So far we have no nonzero examples, which isn't great.

<u>Lemma</u> $H^n(B;\mathbb{Z}) \longrightarrow H^n(B;\mathbb{Z}_2)$ carries $e(\xi)$ to $w_n(\xi)$.

<u>PF</u>

$$\begin{array}{ccc} e(\xi) & = & \phi^{-1}(u \cup u) \\ \downarrow & & \downarrow \\ w_n(\xi) & = & \phi^{-1}(Sq^n(u)) \end{array}$$

<u>Lemma</u> The Euler class of a Whitney sum is $e(\xi_1 \oplus \xi_2) = e(\xi_1)e(\xi_2)$

The Euler class of a Cartesian product is $e(\xi_1 \times \xi_2) = e(\xi_1) \times e(\xi_2)$

<u>PF</u> For a reason choice of orientation convention, the fundamental class of the product is

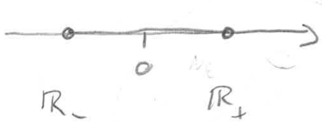
$$u(\xi_1 \times \xi_2) = (-1)^{mn}(u(\xi_1) \times u(\xi_2))$$

Now $H^{m+n}(E_1 \times E_2, (E_1 \times E_2)_0) \longrightarrow H^{m+n}(E_1 \times E_2) \simeq H^{m+n}(B_1 \times B_2)$

$e(V \times W) = (-1)^{mn} e(V)e(W)$ $\}$ Sign doesn't matter; if m or n is odd this is an element of order 2.

<u>But note</u> $e(\xi_i)$ could be a zero divisor, i.e., this might not be solvable.

<u>Proof of the Theorem</u> With field coefficients Let $e' \in H^1(\mathbb{R}, \mathbb{R}_0; \mathbb{Z}_2)$ be the nonzero element. We have a long exact sequence arising from the triple

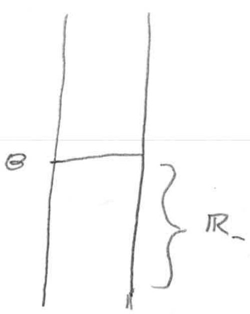


$$(B \times \mathbb{R}, B \times \mathbb{R}_0, B \times \mathbb{R}_-)$$

$$\overset{0}{\tilde{\hat H}^{j+1}(B\times\mathbb{R}, B\times\mathbb{R}_-)} \longleftarrow \hat H^{j+1}(B\times\mathbb{R}_0, B\times\mathbb{R}_-)$$

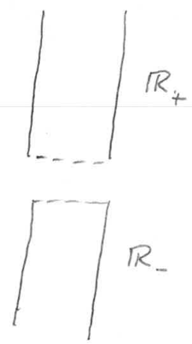
$$\underset{\hat H^j(B)}{\tilde{\hat H}^{j}(B\times\mathbb{R}_0, B\times\mathbb{R}_-)} \longleftarrow \overset{0}{\hat H^{j}(B\times\mathbb{R}, B\times\mathbb{R}_-)}$$

$\tilde{\hat H}^j(B\times\mathbb{R}, B\times\mathbb{R}^-) = 0$



$\hat H^j(B\times\mathbb{R}_0, B\times\mathbb{R}_-) \simeq \hat H^j(B\times\mathbb{R}_+)$

$\simeq \hat H^j(B)$



We conclude

$\hat H^{j+1}(B\times\mathbb{R}, B\times\mathbb{R}_-) \simeq \hat H^j(B)$

If $B' \subseteq B$ open, look at, For $y \in H^j(B\times B')$, $y\times e' \in H^{j+1}(B\times\mathbb{R}, B'\times\mathbb{R}\cup B\times\mathbb{R}_0)$,

Exercise Using the Five lemma, we see the cross product is an isomorphism $H^j(B, B') \longrightarrow H^{j+1}(B\times\mathbb{R}, (B'\times\mathbb{R})\cup(B\times\mathbb{R}_0))$

$\Rightarrow y \longrightarrow y\times e' \longrightarrow y\times \underbrace{e'\times e'}_{e^2} \longrightarrow \cdots$ is an isomorphism. $H^j(B) \longrightarrow H^{j+n}(B\times\mathbb{R}^n, B\times\mathbb{R}^n_0)$

Thom Isomorphism

Case I Say $E = B\times\mathbb{R}^n$. Then $H^n(E, E_0) \simeq H^n(B\times\mathbb{R}^n, B\times\mathbb{R}^n_0) \simeq H^0(B)$
$$\downarrow$$
$$B$$

by the discussion above. Let $1 \in H^0(B)$ correspond to $u = 1\times e^n$.

Then $H^i(B\times\mathbb{R}^n) \longrightarrow H^i(B\times\mathbb{R}^n, B\times\mathbb{R}^n_0)$

$y\times 1 \longmapsto (y\times 1)\cup u = (y\times 1)\cup(1\times e^n) = y\times e^n$,