Lecture 24

Last Time The Stiefel-Whitney classes $w_i(\xi) \in H^i(B;\mathbb{Z}_2)$

Stiefel-Whitney numbers & cobordance

Say M^n is smooth. We have $H_n(M;\mathbb{Z}_2) \simeq \mathbb{Z}_2$. Let μ_M be the fundamental homology class. Given any $(r_1, \dots, r_n)$ such that $r_1 + 2r_2 + 3r_3 + \dots + nr_n = n$, we can consider

$\langle w_1(TM)^{r_1} w_2(TM)^{r_2} \cdots w_n(TM)^{r_n}, \mu_M \rangle = w_1^{r_1} \cdots w_n^{r_n}[M]$. This is the Stiefel-Whitney number of M associated to $w_1^{r_1} \cdots w_n^{r_n}$.

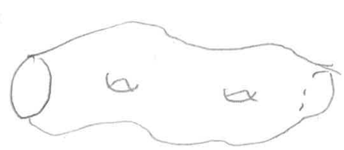
Example $\mathbb{RP}^n$

$w_n = (n+1)a^n \qquad n$ even $\rightsquigarrow w_n[\mathbb{RP}^n] = 1$

$w_1 = (n+1)a \qquad\qquad w_1^n[\mathbb{RP}^n] = 1$

Thm $M = \partial B$ for B smooth, compact $\Leftrightarrow$ all Stiefel-Whitney numbers of M are 0.

Corollary If M_1 and M_2 are cobordant, then they have the same Stiefel-Whitney numbers.



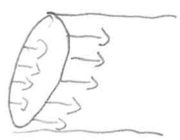
PF $\Leftarrow$ Too hard for today.

$\Rightarrow$

$$\begin{array}{ccc} \mu_B \in H_{n+1}(B, M) & \xrightarrow{\partial} & H_n(M) \\ \mu_B & \longmapsto & \mu_M \end{array}$$

The connecting map takes μ_B to μ_M.

Let $\delta : H^n(M) \to H^{n+1}(B, M)$ be the connecting map in the opposite direction, such that $\langle v, \partial \mu_B \rangle = \langle \delta v, \mu_B \rangle$. Now $TM \subseteq TB|_M$ breaks up as $TM \oplus \varepsilon^1 = TB|_M$. So $w(TM)|_M = w(TM)$. But



Here we use B cpt: $\exists$ a Euclidean metric on TB

$$\begin{array}{ccccc} H^n(B) & \longrightarrow & H^n(M) & \xrightarrow{\delta} & H^{n+1}(B, M) \\ w_1^{r_1} \cdots w_n^{r_n}(TB) & \longmapsto & w_1^{r_1} \cdots w_n^{r_n}(TM) & \longmapsto & 0 \end{array}$$

We see that $\delta(w_1^{r_1} \cdots w_n^{r_n}(TM)) = 0 \Rightarrow \langle 0, \mu_B \rangle = \langle w_1^{r_1} \cdots w_n^{r_n}, \overset{\mu_M}{\overset{\parallel}{\partial \mu_B}} \rangle$

$= w_1^{r_1} \cdots w_n^{r_n}[M]$.

Cohomology of the Grassmanian

We've just seen $H^*(Gr_n(\mathbb{R}^\infty); \mathbb{Z}_2)$ is very important to real vector bundles. Certainly it contains at least $w_1(\gamma^n), \ldots, w_n(\gamma^n)$.

Claim 1 There are no polynomial relations between the classes $w_i(\gamma^n)$.

Proof Suppose there were. Then by the classification theorem, this relationship would hold for the S-W classes of every single $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ over a paracompact base.

However, we can consider the bundle

$$\begin{array}{c} \gamma_1^1 \times \gamma_1^1 \times \cdots \times \gamma_1^1 \\ \downarrow \\ \mathbb{RP}^1 \times \cdots \times \mathbb{RP}^1 = B \end{array}$$

Now $w = (1+a_1)(1+a_2)\cdots(1+a_n)$, where $H^*(B) = \mathbb{Z}_2[a_1, \ldots, a_n]/(a_i^2)$.

$\Rightarrow w_1 = a_1 + \cdots + a_n$

$w_2 = a_1a_2 + a_1a_3 + \cdots$

$\Big\}$ kth elementary symmetric functions. These are linearly independent. $\square$

Proof of Theorem

We will construct a cell structure on the Grassmanian such that the number of r-cells is exactly the number of partitions of r into n integers (i.e. ways of writing $r = a_1 + \cdots + a_n$). This number is an upper bound on the rank of the cohomology. But this is also the number of monomials $w_1^{r_1} \cdots w_n^{r_n}$ in $\mathbb{Z}_2[w_1, \ldots, w_n]$ which have degree r, via

$$r_1 + 2r_2 + \cdots + nr_n = r \iff r = \underbrace{(r_n)}_{a_1} + \underbrace{(r_n + r_{n-1})}_{a_2} + \cdots + \underbrace{(r_n + \cdots + r_1)}_{a_n}$$

Ergo $H^*(Gr_n(\mathbb{R}^\infty); \mathbb{Z}_2) = \mathbb{Z}_2[w_1(\gamma^n), \ldots, w_n(\gamma^n)]$.

Constructing the cell structure

Start w/ $G_n(\mathbb{R}^m)$.

Consider a standard Flag $\mathbb{R}^0 \subseteq \mathbb{R}^1 \subseteq \ldots \subseteq \mathbb{R}^m$.

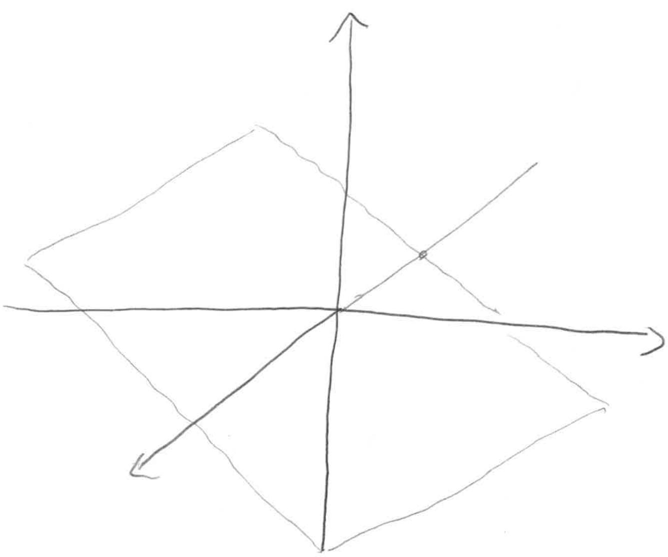
Let X be an n-plane in $\mathbb{R}^m$.

$$0 \leq \underbrace{\dim(X \cap \mathbb{R}^1) \leq \dim(X \cap \mathbb{R}^2)} \leq \ldots \leq \dim(X \cap \mathbb{R}^m)$$

These numbers are equal or differ by 1, which happens n times, by thinking about

$0 \to X \cap \mathbb{R}^{k-1} \to X \cap \mathbb{R}^k \to \mathbb{R}$ via the kth coordinate.

We keep track of the places the dimension changes.



Example

$0 \leq 1 \leq 1 \leq 2$

Schubert Symbol

$(1, 3)$

A Schubert symbol is a sequence of integers $\sigma = (\sigma_1, \ldots, \sigma_n)$ st $1 \leq \sigma_1 < \sigma_2 < \ldots < \sigma_n \leq m$.

For each symbol σ, $e(\sigma) \subseteq G_n(\mathbb{R}^m)$ is the set of n-planes X st $\dim(X \cap \mathbb{R}^{\sigma_i}) = i$ and $\dim(X \cap \mathbb{R}^{\sigma_i - 1}) = i-1$.

<u>Claim</u> $e(\sigma)$ is a cell of dimn $d(\sigma) = (\sigma_1 - 1) + (\sigma_2 - 2) + \dots + (\sigma_n - n)$.

Toward this, consider $H^k = \{(y_1, \dots, y_k, 0, \dots, 0) : y_k > 0\} \subseteq \mathbb{R}^k \subseteq \mathbb{R}^m$.

Then $X \in e(\sigma) \Leftrightarrow \exists$ a basis $x_1, \dots, x_n$ st $x_1 \in H^{\sigma_1}, \dots, x_n \in H^{\sigma_n}$.

X the row space of

$$\begin{array}{cc} \begin{array}{c} \sigma_1 \quad \sigma_2 \\ \downarrow \quad\; \downarrow \end{array} & \\ \begin{bmatrix} \text{———} \; 1 & 0 & 0 & 0 & 0 & 0 \\ \text{—————} & 1 & 1 & 0 & 0 & 0 \\ \vdots & & & & & \\ \text{———————} & & & & 1 & 0 \end{bmatrix} & \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \\ \begin{array}{r} \uparrow \\ \sigma_n \end{array} & \end{array}$$

<u>Lemma</u> Each n-plane $X \in e(\sigma)$ has a unique orthonormal basis $(x_1, \dots, x_n)$ in $H^{\sigma_1} \times \dots \times H^{\sigma_n}$

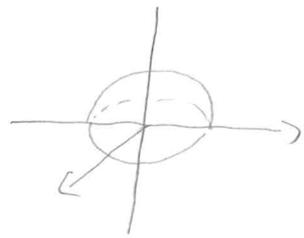
<u>Pf</u> There is a unique choice for $b_1 \in X \cap \mathbb{R}^{\sigma_1}$, given the condition of ending on a positive number. After picking b_1, there is a unique choice for $b_2 \in X \cap \mathbb{R}^{\sigma_2}$, and so on.

<u>Defn</u> Let $e'(\sigma) = V_n(\mathbb{R}^m) \cap (H^{\sigma_1} \times \dots \times H^{\sigma_n})$ be the set of orthonormal n-frames such that each x_i belongs to H^{σ_i}. Similarly let $\bar{e}'(\sigma)$ be $V_n(\mathbb{R}^m) \cap (\bar{H}^{\sigma_1} \times \dots \times \bar{H}^{\sigma_n})$.

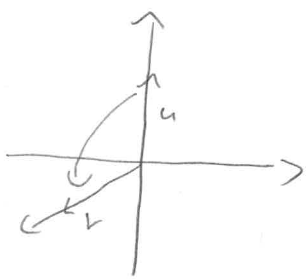
<u>Lemma</u> $\bar{e}'(\sigma)$ is a closed cell of dimn $d(\sigma)$. Moreover the projection $q: V_n(\mathbb{R}^m) \to G_n(\mathbb{R}^m)$ maps its interior $e'(\sigma)$ to $e(\sigma)$ homeomorphically.

Pf An induction on n.

Case I $n=1$ $\quad \bar{e}'(\sigma_1) = \{ x_1 = (x_{11}, \dots, x_{1\sigma_1}, 0, \dots, 0) : |x_1| = 1,\ x_{1\sigma_1} > 0 \}$ is a disk of dimension $\sigma_1 - 1$.



Inductive Step For unit vectors $u, v \in \mathbb{R}^m$, $u \neq -v$, let $T(u,v)$ be a rotation carrying u to v.



- $T(u,v) = id$
- $T(u,v) = T(v,u)^{-1}$
- Everything $\perp$ to the plane of rotation is preserved.

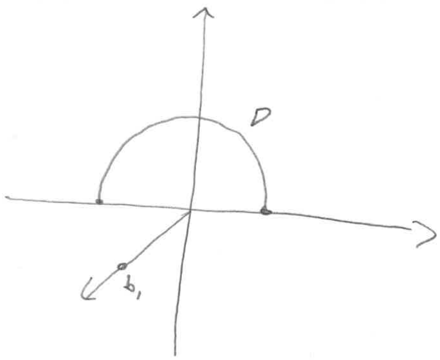
Let $b_i \in H^{\sigma_i}$ be the vector w/ σ_ith coordinate 1 and other coordinates 0. Rotate to a random $(x_1, \dots, x_n) \in H^{\sigma_1} \times \cdots \times H^{\sigma_n}$

$$T = \underbrace{(T(b_n, x_n)) \circ T(b_{n-1}, x_{n-1}) \circ \cdots}_{\substack{\text{Each map here only affects planes not including } x_i \text{ or } b_i. \\ \text{In particular, doesn't move } x_i.}} \circ\, T(b_i, x_i) \circ \underbrace{\cdots \circ T(b_1, x_1)}_{\substack{\text{Each map here only affects } \mathbb{R}^{\sigma_i - 1}. \\ \text{Doesn't move } b_i.}}$$

So $T : \mathbb{R}^m \to \mathbb{R}^m$ has $T(b_i) = x_i$

Let $\sigma_{n+1} > \sigma_n$. Let $D = \{u \in \bar{H}^{\sigma_{n+1}} : b_1 \cdot u = \dots = b_n \cdot u = 0\}$

e.g. $\sigma_1 = 1$, $\sigma_2 = 3$.



D is a closed hemisphere of dimn $\sigma_{n+1} - n - 1$.

<u>Claim</u> $\exists$ a map $F : \bar{e}(\sigma_1, \dots, \sigma_n) \times D \longrightarrow \bar{e}'(\sigma_1, \dots, \sigma_{n+1})$

$$((x_1, \dots, x_n), u) \longmapsto (x_1, \dots, x_n, Tu)$$

This moves u to be orthogonal to $x_1, \dots, x_n$ instead of $b_1, \dots, b_n$.

Notice that $\cdot\ x_i \cdot Tu = Tb_i \cdot Tu = b_i \cdot u = 0$ for $i \le n$

$\cdot\ Tu \cdot Tu = u \cdot u = 1$ and lies in $\bar{H}^{\sigma_{n+1}}$ since it is the same as u except for a change in $\mathbb{R}^{\sigma_n}$.
(In particular, (σ_{n+1})st coordinate is still positive and further coordinates are still zero.)

<u>Exercise</u> F has a cts inverse.

So we have $\bar{e}'(\sigma_1, \dots, \sigma_{n+1}) \cong \bar{e}'(\sigma_1, \dots, \sigma_n) \times D$ is a closed cell of dimn $d(\sigma)$. Similarly $e'(\sigma) = \bar{e}'(\sigma)^\circ$ is an open cell.

Now consider $q|_{e'(\sigma)} : e'(\sigma) \longrightarrow e(\sigma)$. We already know this

$$\begin{array}{cc} \cap\!| & \cap\!| \\ V_n(\mathbb{R}^m) & Gr_n(\mathbb{R}^m) \end{array}$$

map is injective. Moreover if $(x_1, \dots, x_n)$ in $\bar{e}'(\sigma) - e'(\sigma)$, then $X = q(x_1, \dots, x_n)$ has a vector x_i in $\partial(\bar{H}^{\sigma_i}) \Rightarrow X \notin e(\sigma)$ since $\dim(X \cap \mathbb{R}^{\sigma_i - 1}) \geq i$. Now let $A \subset e'(\sigma)$ be closed in $e'(\sigma)$. Then $\bar{A} \cap e'(\sigma) = A$, and $\bar{A} \subseteq \bar{e}'(\sigma)$ is compact, so $q(\bar{A})$ is closed. This implies that $q(\bar{A}) \cap e(\sigma) = q(A)$ is closed in $e(\sigma)$. Ergo $q : e'(\sigma) \to e(\sigma)$ homeomorphically.

Thm The $\binom{m}{n}$ sets $e(\sigma)$ are the cells of a CW structure for $Gr_n(\mathbb{R}^m)$. Taking $m \to \infty$ we get a cell structure for $Gr_n(\mathbb{R}^\infty)$.

Pf Since $\bar{e}'(\sigma)$ is compact, $q(\bar{e}'(\sigma))$ is $\bar{e}(\sigma)$, so every n-plane in the bdry has a basis $(x_1, \dots, x_n)$ in $\bar{e}'(\sigma) - e'(\sigma)$. Now $(x_1, \dots, x_n)$ are orthonormal and $x_i \in \mathbb{R}^{\sigma_i} \Rightarrow \dim(X \cap \mathbb{R}^{\sigma_i}) \geq i \;\; \forall\, i$. So if $(\tau_1, \dots, \tau_n)$ is the Schubert symbol for X has $\tau_1 \leq \sigma_1, \dots, \tau_n \leq \sigma_n$. One x_i is in $\mathbb{R}^{\sigma_i - 1} \Rightarrow \tau_i < \sigma_i \Rightarrow d(\tau) < d(\sigma)$. $\square$

How many cells?

$(\sigma_1, \dots, \sigma_n)$ $\quad d(\sigma) = r \quad$ $(\sigma_1 - 1) + \dots + (\sigma_n - n) = r$

$$\Big\updownarrow$$

$(\underbrace{\sigma_1 - 1}_{i_1}, \dots, \underbrace{\sigma_n - n}_{i_s})$ $\quad s \leq n$

$i_s \leq m-n$

We see that the # of cells is the # of partitions of r into at most n integers each $\le (m-n)$.