# Lecture 23

Last time $\left\{ \begin{array}{l} \text{Real } n\text{-plane bundles} \\ \text{over } B \end{array} \right\} \Big/ \text{isomorphism} \longleftrightarrow \langle B, \underbrace{G_n(\mathbb{R}^\infty)}_{BO(n)} \rangle$

Given $c \in H^*(G_n(\mathbb{R}^\infty); \Lambda)$ and $\begin{array}{c} E \\ \downarrow \\ B \end{array} \leftrightarrow F: B \to G_n(\mathbb{R}^\infty)$, we have

$F^*(c) \in H^i(B; \Lambda)$ for each $\Lambda$.

First Set

The Stiefel-Whitney Classes

Many constructions; usually first introduced via axiomatic description.

1 Existence To every real bundle $\xi \left\{ \begin{array}{c} E \\ \downarrow \\ B \end{array} \right.$ there corresponds

a sequence of cohomology classes $w_i(\xi) \in H^i(B; \mathbb{Z}_2)$

with $w_0 = 1 \in H^0(B; \mathbb{Z}_2)$ and $w_i(\xi) = 0$ for $i > n$ if $V$ is an $n$-plane bundle.

2 Naturality If $f: B_1 \to B_2$ is covered by a bundle map $E_1 \to E_2$, then $w_i(\xi_1) = F^* w_i(\xi_2)$.

$$\begin{array}{ccc} E_1 & \longrightarrow & E_2 \\ \downarrow & & \downarrow \\ B_1 & \xrightarrow{F} & B_2 \end{array}$$

3 Whitney Product Theorem

$$\begin{tikzcd} E_1 \arrow[dr] & & E_2 \arrow[dl] \\ & B & \end{tikzcd} \qquad w_k(\xi_1 \oplus \xi_2) = \sum_{i=0}^{k} w_i(\xi_1)\, w_{k-i}(\xi_2)$$

4 $\gamma_1^1$ has nontrivial $w_1$.

$$\begin{tikzcd} \gamma_1^1 \arrow[d] \\ \mathbb{RP}^1 \end{tikzcd}$$

For the moment, assume this is possible.

Consequences

1. $\xi_1 \simeq \xi_2 \Rightarrow w_i(\xi_1) = w_i(\xi_2)$
2. $\xi$ trivial $\Rightarrow w_i(\xi) = 0$ for $i > 0$.
3. $\xi$ has a nowhere zero section $\Rightarrow w_n(\xi) = 0$.

Because of Whitney product, it's convenient to write
$w(\xi) = 1 + w_1(\xi) + w_2(\xi) + \cdots \in H^{\Pi}(B; \mathbb{Z}_2)$.

Formal infinite sums
$a_0 + a_1 + a_2 + \cdots$

Note that this is a group with inverses as long as $a_0 = 1$, as you'd expect.

$$(1 + a_1 + a_2 + a_3 + \cdots)(1 + b_1 + b_2 + \cdots)$$

$a_1 + b_1 = 0 \rightsquigarrow a_1 = b_1$

$a_2 + a_1 b_1 + b_2 = 0 \rightsquigarrow b_2 = a_2 + a_1^2$

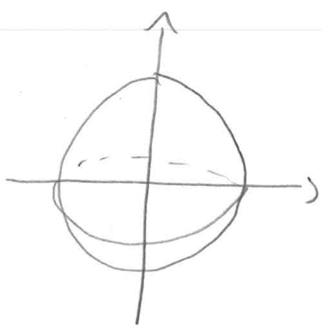
$$\bar{a} = [1 + (a_1 + a_2 + \cdots)^{-1}]$$

$$= 1 - (a_1 + a_2 + a_3 \cdots) + (a_1 + a_2 + a_3 + \cdots)^2 + \cdots$$

<u>Whitney Duality</u> $M \subseteq \mathbb{R}^k$ smooth

$$\bar{w}(TM) = w(NM)$$

<u>Trivial for stably trivial bundles</u>



$$\xi \oplus \varepsilon^k = \varepsilon^n$$

$$w(\xi) \cdot w(\varepsilon^k) = w(\varepsilon^n)$$

$$w(\xi)(1) = 1$$

$$\Rightarrow w(\xi) = 1$$

<u>Makes sense</u>: that

$$\begin{tikzcd} TS^n \arrow[r] \arrow[d] & T\mathbb{R}P^n \arrow[d] \\ S^n \arrow[r] & \mathbb{R}P^n \end{tikzcd}$$

has

$$\tilde{H}^*(S^n; \mathbb{Z}_2) \longleftarrow \tilde{H}^*(\mathbb{R}P^n; \mathbb{Z}_2)$$

$$\left.\begin{array}{c} \mathbb{Z}_2 \\ 0 \\ \vdots \\ 0 \\ \mathbb{Z}_2 \end{array}\right\} \xleftarrow{\;0\;} \mathbb{Z}_2[a]/(a^{n+1})$$

<u>By</u> Axiom 4, $w_1(\gamma_1^1) \neq 0$, so the Stiefel-Whitney class is $1 + a$.

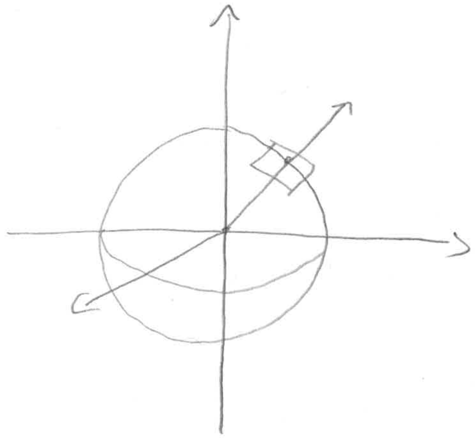
Also we have

$$\begin{tikzcd} \gamma_1^1 \arrow[r] \arrow[d] & \gamma_n^1 \arrow[d] \\ \mathbb{R}P^1 \arrow[r, hook] & \mathbb{R}P^n \end{tikzcd}$$

$w_1(\gamma_n^1) = a$.

$\rightsquigarrow w(\gamma_n^1) = 1 + a$.

What about the perpendicular bundle?



$$w(\gamma^\perp) = \overline{1+a}$$
$$= 1 + a + a^2 + \dots + a^n$$

What about $T(\mathbb{R}P^n)$? Recall we showed

$T\mathbb{R}P^n \oplus \varepsilon^1 \simeq (\gamma^1_n)^{\oplus n+1}$. So we have

$$w(T\mathbb{R}P^n) = (1+a)^{n+1}$$
$$= 1 + \binom{n+1}{1} a + \dots + \binom{n+1}{n} a^n$$

For tangent bundles we sometimes just put the name of the manifold.

Corollary $w(\mathbb{R}P^n) = 1$

Applications

① Division Algebras

When does there exist $q : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ bilinear w/ no zero divisors? $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

Claim Existence of such a map $\Rightarrow$ $\mathbb{R}P^{n-1}$ parallelizable.

PF Let $b_1, \dots, b_n$ be a basis for $\mathbb{R}^n$. Note that

$y \mapsto q(y, b_1)$ is an isomorphism from $\mathbb{R}^n$ to itself. Ergo

$v_i(q(y, b_1)) = q(y, b_i)$ is a linear map $v_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with

$v_1(x) = x$ and the remaining $v_i$ are lin indep. Indeed,

$v_2, \dots, v_n$ are $n-1$ linearly independent sections of $T\mathbb{R}P^{n-1}$

via $T\mathbb{R}P^{n-1} \simeq \mathrm{Hom}(\gamma^1_{n}, \gamma^\perp)$ and $\bar{v}_i : L \longrightarrow L^\perp$, where

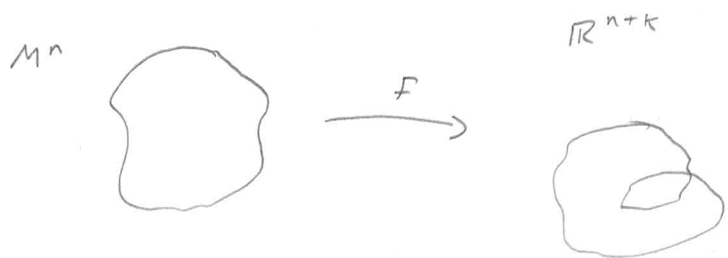
$x \longmapsto \bar{v}_i(x)$

$\bar{v}_i(x)$ is the image of $v_i(x)$ under $\mathbb{R}^n \longrightarrow L^\perp$. $\bar{v}_1 = 0$, but

$\bar{v}_2, \dots, \bar{v}_n$ are lin indep.

This rules out any option where $\mathbb{R}^n$ has $n$ not a power of 2.

## 2 Immersions




$F^*(T\mathbb{R}^{n+k}) = TM \oplus \nu$ ← k-dimensional

If $M$ immerses into $\mathbb{R}^{n+k}$, then $\bar{w}(TM)$ has only terms up to degree $k$!

# Examples

① $w(\mathbb{RP}^{11}) = (1+a)^{12} = 1+a^4+a^8$

$\bar{w} = 1+a^4$ } Doesn't immerse in anything smaller than $\mathbb{R}^{15}$

② $n = 2^r$

$w(\mathbb{RP}^n) = 1+a+a^n$

$\bar{w} = 1+a^{n-1}$

} Doesn't immerse in anything smaller than $\mathbb{R}^{2n-1}$. But does immerse there, by Whitney immersion.

## One Construction of the Stiefel-Whitney Classes

Recall Leray-Hirsch:

Let
$$\begin{array}{ccc} F & \hookrightarrow & E \\ & & \downarrow \\ & & B \end{array}$$
a fibre bundle. Suppose $H^*(F;R)$ is a free, finitely-generated $R$-module, and there exist classes $h_1, \dots, h_k$ in $H^*(E;R)$ whose restrictions $\iota^*(h_1), \dots, \iota^*(h_k)$ are a basis for $H^*(F;R)$ $\forall$ fibre. Then $H^*(E;R) \xrightarrow{\sim} H^*(B;R) \otimes_R H^*(F;R)$.

Indeed $H^*(E;R)$ is a module over $H^*(F;R)$ w/ coefficients $H^*(B;R)$.

Say $\begin{array}{c} V \\ \downarrow p \\ B \end{array}$ is a vector bundle. We can consider its projettivization

$P(V)$, a fibre bundle with fibre above any point $b$ consisting of lines in $F_b$.

$$\begin{tikzcd} \mathbb{RP}^{n-1} \arrow[r, hook] & P(V) \arrow[d, "\pi"] \\ & B \end{tikzcd}$$

We have a universal subbundle on $P(V)$: we can consider every vector in the line.

$$\begin{tikzcd} \gamma_n^1 \arrow[r] \arrow[d] & S = \{(x,v) : x \in P(V),\ v \in x\} \arrow[d] & \\ \mathbb{RP}^{n-1} \arrow[r] & P(V) \arrow[r] & B \end{tikzcd}$$

Let $h$ be the cohomology class associated to this line bundle (line bundles are $H^2(X; \mathbb{Z}_2)$). Then $h$ restricts to exactly the cohomology class associated to the canonical line bundle on $\mathbb{RP}^{n-1}$, or $a$. And $1, h, h^2, \ldots, h^{n-1}$ always restrict to a basis for the cohomology of each fibre $\mathbb{RP}^{n-1}$.

What about $h^n$? It can be written

$$h^n = w_n \cdot 1 + w_{n-1} \cdot h + \cdots + w_1 \cdot h^{n-1}$$ for some $w_1, \ldots, w_n$. These are the Stiefel-Whitney classes!