Last time

Real n-plane bundles over $\mathcal{O}_n$ isomorphism

Given $c \in H^*(G_n(\mathbb{R}^\infty); \Lambda)$ and $E \rightarrow \mathcal{O} \rightarrow G_n(\mathbb{R}^\infty)$, we have $f^*: E \rightarrow G_n(\mathbb{R}^\infty)$.

$f^*(c) \in H^i(\mathcal{O}; \Lambda)$ for each $\Lambda$.

First Set

The Stiefel-Whitney classes

Many constructions; usually first introduced via axiomatic description.

1. Existence
   To every real bundle $E \rightarrow \mathcal{O}$ there corresponds a sequence of cohomology classes $w_i(E) \in H^i(\mathcal{O}; \mathbb{Z}_2)$ with $w_0 = 1 \in H^0(\mathcal{O}; \mathbb{Z}_2)$ and $w_i(E) = 0$ for $i > n$ if $V$ is an $n$-plane bundle.

2. Naturality
   If $F: \mathcal{O}_1 \rightarrow \mathcal{O}_2$ is covered by a bundle map $E_1 \rightarrow E_2$, then $w_i(F) = F^*w_i$. $E_1 \rightarrow E_2$
Whitney Product Theorem

\[ \forall \varepsilon_1, \varepsilon_2 \quad w_k (\varepsilon_1 \oplus \varepsilon_2) = \sum_{i=0}^{k} w_i (\varepsilon_1) w_{k-i} (\varepsilon_2) \]

\( \text{RIP} \)

For the moment, assume this is possible.

Consequences

1. \( \varepsilon_1 \cong \varepsilon_2 \implies w_i (\varepsilon_1) = w_i (\varepsilon_2) \)
2. \( \varepsilon \text{ trivial} \implies w_i (\varepsilon) = 0 \) for \( i > 0 \),
3. \( \varepsilon \) has a nowhere zero section \( \implies w_n (\varepsilon) = 0 \).

Because of Whitney product, it's convenient to write

\[ w (\varepsilon) = 1 + w_1 (\varepsilon) + w_2 (\varepsilon) + \cdots \in H^\bullet (B; \mathbb{Z}) \]

\( n \) is a formal infinite sum

\[ a_0 + a_1 x + a_2 x^2 + \cdots \]

Note that this is a group with inverses as long as \( a_0 = 1 \)
as you'd expect,

\[ (1 + a_1 x + a_2 x^2 + \cdots) (1 + b_1 x + b_2 x^2 + \cdots) \]

\[ a_1 = b_1, \]

\[ a_2 + a_1 b_1 + b_2 = 0 \implies b_2 = a_2 + a_1^2 \]
\[ \bar{a} = \left( 1 + (a_1 + a_2 + \cdots) \right)^{-1} \]

\[ = 1 - (a_1 + a_2 + a_3 + \cdots) + (a_1 + a_2 + a_3 + \cdots)^2 + \cdots \]

**Whitney Duality** \( M \in \mathbb{R}^k \) smooth

\[ \bar{w}(TM) = w(NM) \]

**Trivial for stably trivial bundles**

\[ \mathbb{E} \oplus \mathbb{E}^k = \mathbb{E}^n \]

\[ w(\mathbb{E}) \cdot w(\mathbb{E}^k) = w(\mathbb{E}^n) \]

\[ w(\mathbb{E})(1) = 1 \]

\[ \Rightarrow w(\mathbb{E}) = 1 \]

**Makes sense!** That \( T\mathbb{S}^n \to T\text{IRIP}^n \) has \( H^*(\mathbb{S}^n, \mathbb{Z}_2) \to H^*(\text{IRIP}^n, \mathbb{Z}_2) \)

\[ \mathbb{Z}_2 \otimes \mathbb{Z}_2 [q] / (q^{n+1}) \]

By Axiom 4, \( w_1(\mathbb{E}_1') = 0 \), so the Stiefel-Whitney class is \( 1 + q \).

Also we have \( \mathbb{E}_1' \to \mathbb{E}_n' \)

\[ w_1(\mathbb{E}_n') = q' \]

\[ w_0(\mathbb{E}_n') = 1 + q \]
What about the perpendicular bundle?

\[ w(\mathbb{R}^n) = \overline{1+a} = 1 + a + a^2 + \ldots + a^n \]

What about \( T(\mathbb{R}P^n) \)? Recall we showed:

\[ T(\mathbb{R}P^n) \cong (\mathbb{R}^n)^\oplus n+1 \]

So we have

\[ w(\mathbb{R}P^n) = (1+a)^{n+1} = 1 + \binom{n+1}{1}a + \ldots + \binom{n+1}{n}a^n \]

For tangent bundles we sometimes just put the name of the manifold.

**Corollary** \( w(\mathbb{R}P^n) = 1 \)

**Applications**

1. **Division Algebras**

   When does there exist \( q : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) bilinear w/ no zero divisors? \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \)

   **Claim** Existence of such a map \( \Rightarrow \mathbb{R}P^{n-1} \) parallelizable.
PF Let \( b_1, \ldots, b_n \) be a basis for \( \mathbb{R}^n \). Note that

\[ \text{if } y \mapsto g(y, b_i) \text{ is an isomorphism from } \mathbb{R}^n \text{ to itself, then } \]

\[ v_i (g(y, b_i)) = g(y, b_i) \]

is a linear map \( v_i : \mathbb{R}^n \rightarrow \mathbb{R}^n \) with \( v_i (x) = x \) and the remaining \( v_i \) are lin indep. Indeed, \( v_2, \ldots, v_n \) are \( n-1 \) linearly independent sections of \( T\mathbb{R}P^{n-1} \) via \( T\mathbb{R}P^{n-1} \cong \text{Hom}(\mathbb{R}^n, \mathbb{R}^n) \) and \( \tilde{v}_i : \ell \rightarrow L^1 \), where \( x \mapsto \tilde{v}_i (x) \)

\( \tilde{v}_i (x) \) is the image of \( v_i (x) \) under \( \mathbb{R}^n \rightarrow L^1 \). \( \tilde{v}_i = 0 \), but \( \tilde{v}_2, \ldots, \tilde{v}_n \) are lin indep.

This rules out any option where \( \mathbb{R}^n \) has \( n \) not a power of 2.

(2) **Immersions**

\[ M^n \quad \xrightarrow{f} \quad \mathbb{R}^{n+k} \]

\[ F^\ast(T\mathbb{R}^{n+k}) = TM \oplus R^{k-\text{dimensional}} \]

If \( M \) immerses into \( \mathbb{R}^{n+k} \), then \( \tilde{w}(TM) \) has only terms up to degree \( k! \).
Examples

1. \( w(R^{11}) = (1+a)^{12} = 1+a^4 + a^8 \)

   \( \bar{w} = 1+a^4 \) Doesn't immerse in anything Smaller than \( \mathbb{R}^{15} \)

2. \( n = 2^n \)

   \( w(R^{2^n}) = 1 + a + a^n \)

   \( (1) \bar{w} = 1 + a^{n-1} \) \( \text{Doesn't immerse in anything Smaller than } \mathbb{R}^{2^{n-1}}. \text{ But does immerse there, by Whitney immersion.} \)

One Construction of the Stiefel–Whitney Classes

Recall Leray–Hirsch:

Let \( F \to E \) a Fibre bundle. Suppose \( H^*(F; \mathbb{R}) \) is a free, finitely-generated \( \mathbb{R} \)-module, and there exist classes \( h_1, \ldots, h_k \) in \( H^*(E; \mathbb{R}) \) whose restrictions \( c^*(h_i) \) are a basis for \( H^*(F; \mathbb{R}) \) \& fibre. Then \( H^*(E; \mathbb{R}) \to H^*(E; \mathbb{R}) \otimes H^*(F; \mathbb{R}). \)

Indeed \( H^*(E; \mathbb{R}) \) is a module over \( H^*(F; \mathbb{R}) \) with coefficients \( H^*(F; \mathbb{R}). \)
Say $V$ is a vector bundle, we can consider its projectivization $P(V)$, a fibre bundle with fibre above any point $b$ consisting of lines in $F_b$.

$$
\mathbb{R}P^{n-1} \longrightarrow P(V) \longrightarrow B
$$

We have a universal subbundle on $P(V)$: we can consider every vector in the line,

$$
\mathbb{C}^n \longrightarrow S = S(x, y); x \in P(V), y \in x\mathbb{C}
$$

$$
\mathbb{R}P^{n-1} \longrightarrow P(V) \longrightarrow B
$$

Let $h$ be the cohomology class associated to this line bundle (line bundles are $H^2(x, \mathbb{Z})$). Then $h$ restricts to exactly the cohomology class associated to the canonical line bundle on $\mathbb{R}P^{n-1}$, or $a$. And $h, h^2, \ldots, h^{n-1}$ always restrict to a basis for the cohomology of each fibre $\mathbb{R}P^{n-1}$.

What about $h^n$? It can be written

$$
h^n = w_0 + w_{n-1} h + \ldots + w_n h^{n-1}
$$

for some $w_0, \ldots, w_n$. These are the Stiefel-Whitney classes.