Lecture 22

Last Time

Propn Let $E \xrightarrow{p} B$ be a vector bundle over a compact (Hausdorff) base. Then there is a map $F : B \longrightarrow Gr_n(\mathbb{R}^{n+k})$ such that

$$\begin{tikzcd}
E \simeq F^*(\gamma_n(\mathbb{R}^{n+k})) \arrow[r, "\tilde{F}"] \arrow[d] & \gamma_n(\mathbb{R}^{n+k}) \arrow[d] \\
B \arrow[r, "F"'] & Gr_n(\mathbb{R}^{n+k})
\end{tikzcd}$$

Reminder of proof Suffices to define a map $\hat{F} : E \longrightarrow \mathbb{R}^{n+k}$ which is cts and maps each fibre to an n-dimensional subspace of $\mathbb{R}^n$.

We did this by picking trivializations $h_i : p^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^n$, for $U_1, \dots, U_r$, then picking functions $\lambda_i : B \to \mathbb{R}$ which were 1 on W_i, 0 outside U_i,

could be used to extend h_i to all of E.

$$\begin{cases} h_i'(e) = 0 & \text{if } p(e) \notin U_i \\ h_i'(e) = \lambda_i(p(e))\, h_i(e) & \text{if } p(e) \in U_i \end{cases}$$

Then we can use $\hat{F} : V \longrightarrow \mathbb{R}^n \oplus \dots \oplus \mathbb{R}^n$.

$$e \longmapsto (h_1'(e), \dots, h_r'(e))$$

$\square$

What about noncompact bases?

The <u>universal bundle</u> is

$$\begin{array}{c} \gamma^n = \{(X, v) : X \in G_n(\mathbb{R}^\infty),\ v \in X\} \\ \downarrow \\ BO(n) \simeq G_n(\mathbb{R}^\infty) \end{array}$$

<u>Propn</u> Any n-dim'l vector bundle $E \longrightarrow B$ over a paracompact base is the pullback of a map

$$\begin{tikzcd} E \simeq F^*(\gamma^n) \arrow[r] \arrow[d] & \gamma^n \arrow[d] \\ B \arrow[r, "F"'] & G_n(\mathbb{R}^\infty) \end{tikzcd}$$

<u>Follows from</u>

<u>Lemma</u> For any fibre bundle E over a paracompact base B, $\exists$ a locally finite covering of B by countably many open sets $U_1, U_2, \ldots$, so that $E|_{U_i}$ is trivial for each i.

<u>Pf</u> Pick any open cover $\{U_\alpha\}$ such that $E|_{U_\alpha}$ is trivial. By paracompactness, we can replace w/ locally finite $\{X_\alpha\}$ such that $E|_{X_\alpha}$ is trivial. Pick $\{W_\alpha\}$ another cover w/ $\overline{W_\alpha} \subseteq U_\alpha$. [Paracompact spaces are normal, exercise]. Let $\lambda_\alpha : B \to \mathbb{R}$ be cts st λ_α is 1 on $\overline{W_\alpha}$ and 0 outside X_α. If S is a finite subset of α, let $U(S) \subseteq B$ denote the set of $b \in B$ for which

$$\min_{\alpha \in S} \lambda_\alpha(b) > \max_{\alpha \notin S} \lambda_\alpha(b)$$

Let $U_k = \bigcup_{|s|=k} V(s)$. Then $B = U_1 \cup U_2 \cup U_3 \cup \cdots$.

This is enough to run the same argument & prove the proposition.

<u>Remark</u> Local triviality of the universal bundle follows essentially the same proof as for the canonical bundle, except that one needs to check that if $K_1 \subseteq K_2 \subseteq \cdots$ and $L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots$ are sequences of inclusions of locally cpt spaces, then the direct limit $\varinjlim (K_i \times L_i)$ has the same topology as $(\varinjlim K_i) \times (\varinjlim L_i)$.

<u>Ok, so far</u> Any n-dimensional bundle over a paracompact base has a map

$$\begin{array}{ccc} E \simeq \bar{f}^*(\gamma^n) & \longrightarrow & \gamma^n \\ \downarrow & & \downarrow \\ B & \xrightarrow[\bar{f}]{} & Gr_n(\mathbb{R}^\infty) \simeq BO(n) \end{array}$$

<u>From the hw</u> Homotopic maps induce isomorphic vector bundles

<u>Lemma</u> Any two bundle maps $\tilde{f}, g : \xi \longrightarrow \gamma^n$ are homotopic through vector bundle maps.

<u>Pf</u> Say we have

$$\begin{array}{ccc} E & \xrightarrow{f,g} & \gamma^n \\ p\downarrow & & \downarrow \\ B & \xrightarrow{f,g} & Gr_n(\mathbb{R}^\infty) \end{array}$$

observe that we have a map $\hat{f} : E \longrightarrow \gamma^n \subseteq Gr_n(\mathbb{R}^\infty) \times \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$. Furthermore, $\hat{f}$ deterf via $F(e) = (\hat{F}(\text{Fiber through } e), \hat{F}(e))$. So this is really a question about maps into $\mathbb{R}^\infty$.

We claim $\hat{F} \sim \hat{g}$.

<u>Case I</u> If $\hat{F}(e) \neq -c\hat{g}(e)$, we can just use a linear homotopy $\hat{h}_t(e) = (1-t)\hat{F}(e) + t\hat{g}(e)$. Because $\hat{h}_t(e)$ is never 0, we can use this to define a homotopy $h_t : E \times [0,1] \to \gamma^n$ via $h_t(e) = (\hat{h}_t(\text{Fibre through } e), \hat{h}_t(e))$. To check this is cts, we need to check that the induced map $\bar{h}_t : B \times [0,1] \to Gr_n(\mathbb{R}^\infty)$ is cts. But over any $U \subseteq B$ such that $E|_U$ is trivial w/ $s_1, \dots, s_n$ nowhere dependent sections. Then $\bar{h}$ is a composition of

① $(b,t) \mapsto (\hat{h}_t s_1(b), \dots, \hat{h}_t s_n(b))$ From $U \times [0,1] \to V_n(\mathbb{R}^\infty)$

② Projection $V_n(\mathbb{R}^\infty) \to G_n(\mathbb{R}^\infty)$.

<u>Case II</u> F, g arbitrary. Then consider $d_1 : \gamma^n \to \gamma^n$ be the map carrying the ith basis vector to the $(2i-1)$st and $d_2 : \gamma^n \to \gamma^n$ be the map carrying the ith basis vector to the $2i$th.

Then we have

$$f \sim d_1 \circ f \sim d_2 \circ g \sim g$$ by three applications of Case I.

We conclude that n-dimensional vector bundles over $\mathbb{R}^n$ correspond to homotopy classes of maps $\langle B, Gr_n(\mathbb{R}^\infty)\rangle$

$$\cong$$

$$\langle B, BO(n)\rangle$$

Similarly, complex vector bundles are maps $\langle B, BU(n) \rangle$.

Let's think about this in the Context of vector bundle equivalence.

The set of isomorphism classes of vector bundles over a base B has an operation of addition ($\oplus$) but no natural inverses.

We introduce two new notions of equivalence:

- E_1 and E_2 over B are stably isomorphic if $E_1 \oplus \varepsilon^n \simeq E_2 \oplus \varepsilon^n$ for some n. We write $E_1 \underset{s}{\approx} E_2$
- E_1 and E_2 are similar if $E_1 \oplus \varepsilon^n \simeq E_2 \oplus \varepsilon^m$ for some n, m. We write $E_1 \sim E_2$

Easiest case

$$\tilde{K}^0(B) = \{\text{Equivalence classes } [E] \text{ of vector bundles ove } B \text{ up to } \sim\}$$

Propn If B is compact, $\tilde{K}(B)$ is a group w/ composition given by $\oplus$.

PF The nontrivial thing is the existence of inverses.
Recall that

$$\begin{tikzcd} E \simeq f^*(\gamma^n(\mathbb{R}^{n+k})) \arrow[r] \arrow[d] & \gamma^n(\mathbb{R}^{n+k}) \subseteq Gr_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k} \arrow[d] \\ B \arrow[r, "f"] & Gr_n(\mathbb{R}^{n+k}) \end{tikzcd}$$

We pullback $\gamma_n^{\perp}$ to get a bundle E' such that $E \oplus E' \simeq \mathbb{R}^{n+k}$. $\square$

Similarly

$KO^0(B)$ = {Equivalence classes of vector bundles up to $\simeq_s$ and formal differences of same}

$= \{ [E_1 - E_1'] : [E_1 - E_1'] = [E_2 - E_2'] \text{ if } E_1 \oplus E_2' \simeq_s E_2 \oplus E_1' \}$

Checking this relationship is transitive requires cancellation:
Let B be compact, and let $E_1 \oplus E_2 \simeq_s E_1 \oplus E_3$ for bundles E_i over B. Then $E_2 \simeq_s E_3$. For there is a bundle E_1' st $E_1' \oplus E_1 \simeq \varepsilon^n$, so $\varepsilon^n \oplus E_2 \simeq_s \varepsilon^n \oplus E_3$.

Example $KO^0(pt) = \mathbb{Z}$ $\quad$ $\widetilde{KO}^0(S^0) = \mathbb{Z}$ = {difference in dimension of the two fibres}

$\widetilde{KO}^0(pt) = 0$ $\quad$ $\mathbb{R}^m \quad \mathbb{R}^n$

Similarly $K^0(X)$ = ⟨complex vector bundles up to adding formal differences⟩

$\hat{K}^0(X)$ = ⟨complex vector bundles up to $\sim$⟩

<u>So, where are we?</u>

- Notice that 1-dimensional real bundles are

$$\langle B, G_n(\mathbb{R}^\infty)\rangle = \langle B, \mathbb{RP}^\infty\rangle = H^1(B; \mathbb{Z}_2)$$

- 1-dimensional complex vector bundles are

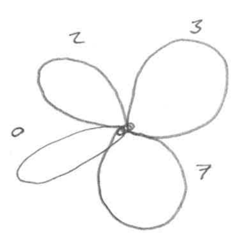
$$\langle B, G_n(\mathbb{C}^\infty)\rangle = \langle B, \mathbb{CP}^\infty\rangle = H^2(B; \mathbb{Z})$$

<u>More generally</u> Let $c \in H^i(G_n; \Lambda)$ be a cohomology class. Then given $\begin{array}{c} E \\ \downarrow \\ B \end{array} \longleftrightarrow f: B \longrightarrow G_n$, we have $f^*(c) \in H^i(B; \Lambda)$ is a <u>characteristic class</u>. [Here G_n is $G_n(\mathbb{R}^\infty)$ or $G_n(\mathbb{C}^\infty)$, and $E \longrightarrow B$ is real or complex as appropriate.]

<u>First Example</u> The Stiefel-Whitney classes.

$$\xi\left\{\begin{array}{c} E \\ \downarrow \\ B \end{array}\right.$$

Look at B^1



$w_1(\xi)$ is the class that evaluates to zero if the bundle restricted to the circle is trivial and 1 if nontrivial.

[Any circle bundle is in $\langle S^1, BO(n)\rangle \simeq \mathbb{Z}_2$, so odd vs. even is the correct thing here.]