Lecture 21

<u>Recall</u> $u : V \to \mathbb{R}$ on finite-dimensional V is <u>quadratic</u> if it can be expressed as $u(v) = \sum l_i(v) l_i'(v)$, where each l_i and l_i' are linear. Such a function determines a bilinear pairing $\langle v, w \rangle = \frac{1}{2}(u(v+w) - u(v) - u(w))$, with $v \cdot v = u(v)$.

We say $u(v)$ is positive definite if $u(v) > 0$ for $v \neq 0$.

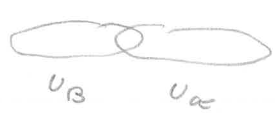
<u>Defn</u> A Euclidean vector space is a vector space w/ a positive definite quadratic function $u : V \to \mathbb{R}$. (This is just another way of defining an inner product space.)

<u>Defn</u> A Euclidean vector bundle is a real vector bundle ξ w/ a continuous $u : E(\xi) \to \mathbb{R}$ st $u|_{F_b}$ is positive definite and quadratic for all $b \in B$.

We could ask for our vector bundle to be trivialized in a way that respects the Euclidean structure, i.e. $f \in O(n)$ instead

$$p^{-1}(U_\beta) \simeq U_\beta \times \mathbb{R}^n \xrightarrow{f} U_\alpha \times \mathbb{R}^n \simeq p^{-1}(U_\alpha)$$

of $GL(n, \mathbb{R})$.



But these notions are the same up to isomorphism

Lemma Let ξ be trivial of dimension n over B, and μ be any Euclidean metric on ξ. Then there exist n sections of ξ which are orthonormal in the sense that sense that $s_i(b) \cdot s_j(b) = \delta_{ij}$ $\forall$ $b \in B$.

Pf Suppose $s_1', \ldots, s_n'$ are n nowhere-dependent sections. Use the Gram-Schmidt process to produce an orthonormal basis $s_1(b), \ldots, s_n(b)$. Since everything in said process depends on $s_1'(b), \ldots, s_n'(b)$ and the inner product, the new sections $s_1, \ldots, s_n$ are continuous.

More generally $O(n)$ is a deformation retract of $GL(n, \mathbb{R})$, so as a general matter Euclidean vector bundles are not, up to isomorphism, different from real vector bundles.

Exercise Over a paracompact base, any vector bundle may be equipped w/ a Euclidean metric.

defn next page

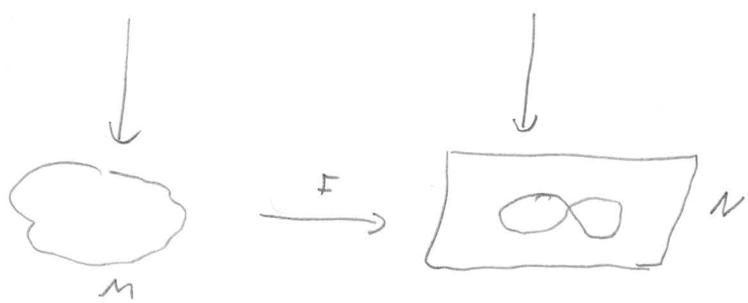
Orthogonal complements

Let $\begin{array}{c}\bar{E} \subseteq E \\ \searrow \swarrow \\ B\end{array}$ be a subbundle. Then if E has a Euclidean metric $\begin{array}{c}\bar{E}^{\perp} \\ \downarrow \\ B\end{array}$ is a subbundle of E, and $\bar{E} \oplus \bar{E}^{\perp} \cong E$.

Local Triviality Let $b_0 \in B$. Let U be a neighborhood of b_0 such that $\tilde{E}, E$ are both trivial over U. Let $s_1, \ldots, s_m$ be orthonormal sections of $\tilde{E}|_U$ and $s_1', \ldots, s_n'$ be orthonormal sections of $E|_U$. We look at the $m \times n$ matrix $[s_i(b_0) \cdot s_j'(b_0)]$. Can assume first m columns are linearly independent after reordering the s_i'. Then $s_1, \ldots, s_m, s_{m+1}', \ldots, s_n'$ are nowhere dependent on $E|_{V_0}$. Use Gram-Schmidt to get something orthogonal $\rightsquigarrow s_1, \ldots, s_m, s_{m+1}, \ldots, s_n$. $\Box$

This justifies our comments about normal bundles last time. More generally, if we have an immersion,

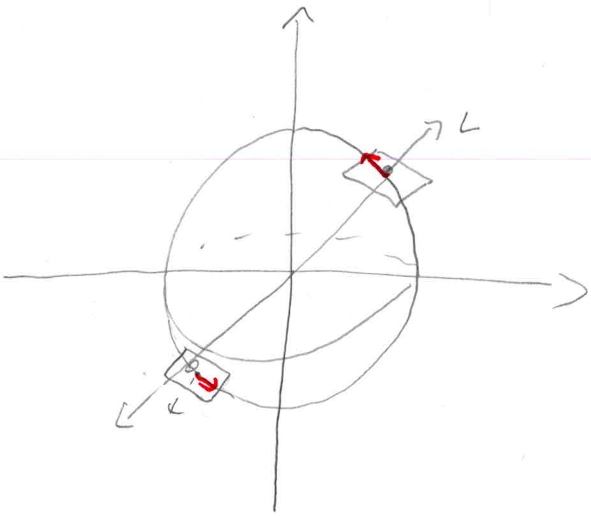
$$TM \oplus \nu_f = f^*(TN) \longrightarrow TN$$



Recall A space B is said to be paracompact if it is Hausdorff and $\forall$ open cover $\{V_\alpha\}$ of B, $\exists$ a locally finite refinement $\{x_\beta\}$ (each x_β is open and contained in V_α for some α). Examples include metric spaces, direct limits of inclusions of cpt spaces. The point here is that paracompact spaces admit partitions of unity.

Let's do a little fiddling w/ new constructions before we start doing classification.

Lemma $T(\mathbb{RP}^n) \simeq \mathrm{Hom}(\gamma_n^1, \gamma^\perp)$, where $\gamma^\perp$ denotes the orthogonal complement to γ_n^1 in $\mathbb{RP}^n \times \mathbb{R}^{n+1}$.



PF An element in $T(\mathbb{RP}^n)$ looks like two elements of TS^n: it's the equivalence class consisting of $\{(x,v), (-x,-v)\}$ where $\|x\|=1$, $x \cdot v = 0$.

We can reduce this to a map

$$\left.\begin{aligned} \ell : L &\longrightarrow L^\perp \\ x &\longmapsto v \end{aligned}\right\} \text{Element in the hom space.}$$

Lemma $T(\mathbb{RP}^n) \oplus \varepsilon^1$ is isomorphic to $(\gamma_n^1)^{\oplus n+1}$

Proof Consider $\mathrm{Hom}(\gamma_n^1, \gamma_n^1)$. This is a line bundle w/ the identity map as a nowhere zero cross section $\rightsquigarrow$ it is a trivial line bundle over $\mathbb{RP}^n$.

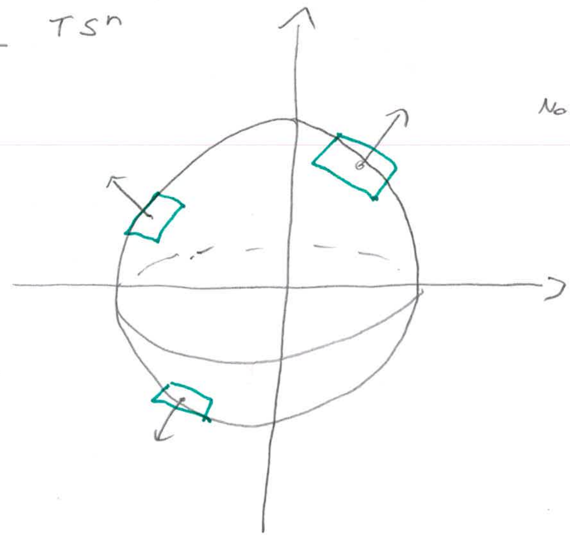
$$\begin{aligned} T(\mathbb{RP}^n) \oplus \varepsilon^1 &\simeq \mathrm{Hom}(\gamma_n^1, \gamma^\perp) \oplus \mathrm{Hom}(\gamma_n^1, \gamma_n^1) \\ &\simeq \mathrm{Hom}(\gamma_n^1, \varepsilon^{n+1}) \\ &\simeq (\mathrm{Hom}(\gamma_n^1, \varepsilon^1))^{\oplus n+1} \end{aligned}$$

$= (\gamma_n^1)^{\oplus n+1}$

We say that the two bundles in the preceding example are <u>similar</u> (they differ by trivial summands). $E_1 \underset{s}{\simeq} E_2$ if $E_1 \oplus \varepsilon^n \simeq E_2 \oplus \varepsilon^m$

A bundle that is similar to the trivial bundle is said to be <u>stably trivializable</u>.

<u>Example</u> TS^n



Normal bundle is trivial

Classification of (real) vector bundles

<u>Last Time</u> Canonical bundle

$$\begin{array}{c} \gamma_n^1 \\ \downarrow \\ \mathbb{R}P^n = G_1(\mathbb{R}^{n+1}) \end{array}$$

More generally, $\gamma^n(\mathbb{R}^{n+k}) \subseteq G_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k}$ is the set

$$\begin{array}{c} \gamma^n(\mathbb{R}^{n+k}) \\ \downarrow \\ G_n(\mathbb{R}^{n+k}) \end{array}$$

of pairs (n-plane in $\mathbb{R}^{n+k}$, vector in said n-plane).

It comes with a vector space structure

$$t_1(x, x_1) + t_2(x, x_2) = (x, t_1 x_1 + t_2 x_2)$$

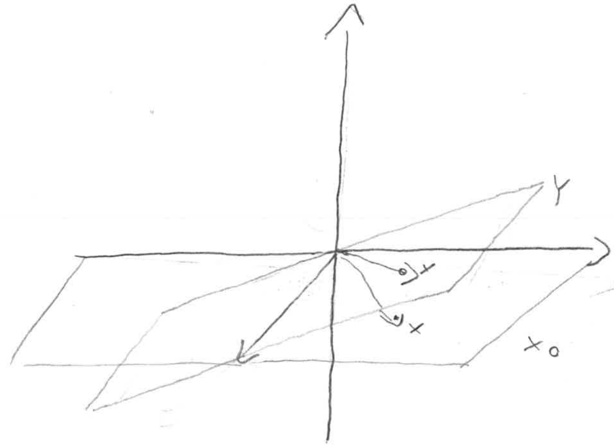
Local triviality of this bundle

Recall that $G_n(\mathbb{R}^{n+k})$ is topologized as a quotient space of $O(n) \curvearrowright V_n(\mathbb{R}^{n+k}) \underset{open}{\subseteq} Mat_{(n+k)\times n}(\mathbb{R})$, so it has the structure of an

$(n+k)n - n(n) = (nk)$-dimensional manifold. We start by constructing a nbhd of any point in $G_n(\mathbb{R}^{n+k})$ homeomorphic to $\mathbb{R}^{nk}$. Let X_0 be an n-plane, and $\mathbb{R}^{n+k} = X_0 \oplus X_0^\perp$. Let $U \subseteq G_n(\mathbb{R}^{n+k})$ be the set of all n-planes Y st orthogonal projection along $\pi: X_0 \oplus X_0^\perp \longrightarrow X_0$ takes Y to X_0. (That is, $Y \cap X_0 = \phi$) This is an open condition. Indeed, each $Y \in U$ is exactly the graph of some $T(Y): X_0 \longrightarrow X_0^\perp$, so $T: \text{Hom}(X_0, X_0^\perp) \simeq \mathbb{R}^{nk}$. Can check this is a homeomorphism.

Now, let $\begin{array}{l} h: \overset{\mathbb{R}^{nk}}{\underset{}{\overset{\text{\rotatebox{90}{$\subseteq$}}}{U}}} \times X_0 \longrightarrow p^{-1}(U) \\ \quad (Y, x) \longmapsto (Y, y) \end{array}$ where y is the preimage of x along the projection π applied to Y.

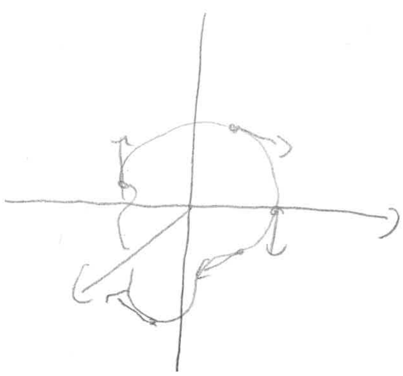
Then $h(Y, x) = (Y, x + T(Y)x)$ and $h^{-1}(Y, y) = (Y, \pi y)$ is continuous.



Why is this bundle interesting?

Example Say that $M \subseteq \mathbb{R}^{n+k}$. We can consider the Gauss map $\bar{g} : M \longrightarrow G_n(\mathbb{R}^{n+k})$

$$x \longmapsto TM_x$$



So

$$\begin{tikzcd}
(x,v) \arrow[r, mapsto] & (TM_x, v) \\[-20pt]
g : TM \arrow[r] \arrow[d] & \gamma^n(\mathbb{R}^{n+k}) \arrow[d] \\
M \arrow[r, "\bar{g}"'] & G_n(\mathbb{R}^{n+k})
\end{tikzcd}$$

$$TM = \bar{g}^*(\gamma^n(\mathbb{R}^{n+k}))$$

This generalizes

Propn For any n-dim'l bundle $E \longrightarrow B$ over a compact base B, there is a map $B \xrightarrow{F} G_n(\mathbb{R}^{n+k})$ such that

$$\begin{tikzcd}
E \simeq F^*(\gamma^n(\mathbb{R}^{n+k})) \arrow[r] \arrow[d] & \gamma^n(\mathbb{R}^{n+k}) \arrow[d] \\
B \arrow[r] & G_n(\mathbb{R}^{n+k})
\end{tikzcd}$$

(Here $k >> 0$)

Proof Pick $U_1, \dots, U_r$ covering B so that $E|_{U_r}$ is trivial. (Finitely many b/c compact). Since B is normal, $\exists$ open sets $X_1, \dots, X_r$ covering B w/ $\overline{X_i} \subseteq U_i$. Similarly we have $W_1, \dots, W_i$ w/ $\overline{W_i} \subseteq X_i$. Let $\lambda_i : B \to \mathbb{R}$ be 1 on $\overline{W_i}$ and 0 outside U_i. Pick a trivialization $\tilde{h}_i : p^{-1}(U_i) \to \mathbb{R}^n$ which is linear on each fibre.

Define $h_i : E \to \mathbb{R}^n$ by

$$\begin{cases} h_i(e) = 0 & \text{if } p(e) \notin U_i \\ h_i(e) = \lambda_i(p(e))\,\tilde{h}_i(e) & \text{for } p(e) \in U_i \end{cases}$$

This is cts and linear on each fibre. Then we can consider

$$\tilde{F} : E \to \underbrace{\mathbb{R}^n \oplus \mathbb{R}^n \oplus \dots \oplus \mathbb{R}^n}_{r \text{ copies}}$$

$$e \longmapsto (h_1(e), \dots, h_n(e))$$

So an isomorphism class of vector bundles is, up to homotopy, a map

$$\begin{tikzcd} E = F^*(\gamma^n(\mathbb{R}^{n+k})) \arrow[r] \arrow[d] & \gamma^n(\mathbb{R}^{n+k}) \arrow[d] \\ B \arrow[r] & G_n(\mathbb{R}^{n+k}) \end{tikzcd}$$