Recall $u : V \to \mathbb{R}$ on finite-dimensional $V$ is quadratic if it can be expressed as $u(v) = \sum l_i(v) l'_i(v)$, where each $l_i$ and $l'_i$ are linear. Such a function determines a bilinear pairing $\langle y, w \rangle = \frac{1}{2} (u(v+w) - u(v) - u(w))$, with $v \cdot v = u(v)$.

We say $u(v)$ is positive definite if $u(v) > 0$ for $v \neq 0$.

**Defn** A Euclidean vector space is a vector space $V$ with a positive definite quadratic function $u : V \to \mathbb{R}$. (This is just another way of defining an inner product space.)

**Defn** A Euclidean vector bundle is a real vector bundle $E$ with a continuous $u : E(\mathbb{E}) \to \mathbb{R}$ s.t.

We could ask for our vector bundle to be trivialized in a way that respects the Euclidean structure, i.e., $\mathbb{E} \subset \mathbb{O}(n)$ instead of $\mathbb{O}(n, \mathbb{R})$.

But these notions are the same up to isomorphism.
Lemma. Let $E$ be trivial of dimension $n$ over $B$, and $m$ be any Euclidean metric on $E$. Then there exist $n$ sections of $E$ which are orthonormal in the sense that $s_i(b) \cdot s_j(b) = 0$ if $i \neq j$ and $b \in B$.

Proof. Suppose $s_1, \ldots, s_n$ are $n$-nowhere-dependent sections. Use the Gram-Schmidt process to produce an orthonormal basis $s_1(b), \ldots, s_n(b)$. Since everything in said process depends on $s_1(b), \ldots, s_n(b)$ and the inner product, the new sections $s_1, \ldots, s_n$ are continuous.

More generally, $O(n)$ is a deformation retract of $GL(n, \mathbb{R})$, so as a general matter Euclidean vector bundles are not, up to isomorphism, different from real vector bundles.

Exercise. Over a paracompact base, any vector bundle may be equipped with a Euclidean metric.

Orthogonal complements

Let $F \subseteq E$ be a subbundle. Then if $E$ has a Euclidean metric $E \cong E^1$ is a subbundle of $E$, and $E \oplus E^1 \cong E$.
Local Triviality: Let $b \in B$. Let $U$ be a neighborhood of $b_0$ such that $\bar{E}, E$ are both trivial over $U$. Let $s_1, \ldots, s_m$ be orthonormal sections of $\bar{E}|_U$ and $s'_1, \ldots, s'_n$ be orthonormal sections of $E|_U$. We look at the $m \times n$ matrix $[s'_i(b_0) \cdot s_j(b_0)]$.

Can assume first $m$ columns are linearly independent after reordering the $s'_i$. Then $s_1, \ldots, s_m, s'_1, \ldots, s'_n$ are nowhere dependent on $E|_U$. Use Gram-Schmidt to get something orthogonal $s_1, \ldots, s_m, s'_1, \ldots, s'_n$. \qed

This justifies our comments about normal bundles last time.

More generally, if we have an immersion,

$$TM \oplus Y_f = f^*(TN) \rightarrow TN$$

Recall: A space $B$ is said to be paracompact if it is Hausdorff and any open cover $\{U_\alpha \}_{\alpha \in \mathcal{A}}$ of $B$ has a locally finite refinement $\{V_\alpha \}_{\alpha \in \mathcal{A}}$ (each $V_\alpha$ is open and contained in $U_\alpha$ for some $\alpha$).

Examples include metric spaces, direct limits of inclusions of cpt spaces. The point here is that paracompact spaces admit partitions of unity.
Let's do a little fiddling w/ new constructions before we start doing classification.

**Lemma** \( T(\mathbb{R}P^n) \cong \text{Hom}(\mathbb{R}^1, \mathbb{R}^1) \), where \( \mathbb{R}^1 \) denotes the orthogonal complement to \( \mathbb{R}^1 \) in \( \mathbb{R}P^n \times \mathbb{R}^{n+1} \).

**Proof** An element in \( T(\mathbb{R}P^n) \) looks like two elements of \( TS^1 \); it's the equivalence class consisting of

\[ (x, v), (-x, -v) \]

where \( xv = 0 \).

We can reduce this to a map

\[ \ell : L \to L^1 \]

Element in the hom space.

**Lemma** \( T(\mathbb{R}P^n) \oplus \mathbb{R}^1 \) is isomorphic to \( (\mathbb{R}^1) \oplus \mathbb{R}^{n+1} \).

**Proof** Consider \( \text{Hom}(\mathbb{R}^1, \mathbb{R}^1) \). This is a line bundle w/ the identity map as a nowhere zero cross section \( \rho \); it is a trivial line bundle over \( \mathbb{R}P^n \).

\[
T(\mathbb{R}P^n) \oplus \mathbb{R}^1 \cong \text{Hom}(\mathbb{R}^1, \mathbb{R}^1) \oplus \text{Hom}(\mathbb{R}^1, \mathbb{R}^1) \\
\cong \text{Hom}(\mathbb{R}^1, \mathbb{R}^1) \\
\cong (\text{Hom}(\mathbb{R}^1, \mathbb{R}^1)) \oplus \mathbb{R}^{n+1}
\]
We say that the two bundles in the preceding example are similar (they differ by trivial summands). \( E_1 \cong E_2 \) if \( E_1 \oplus E^n \cong E_2 \oplus E^n \).

A bundle that is similar to the trivial bundle is said to be stably trivializable.

**Example** \( T\mathbb{S}^n \)

![Diagram of \( T\mathbb{S}^n \) with normal bundle depicted as trivial]

Classification of (real) vector bundles

Last Time: Canonical bundle \( \mathcal{K}_n \)

\[ \mathbb{R}P^n = G_1(\mathbb{R}^{n+1}) \]

More generally, \( \mathcal{K}(\mathbb{R}^{n+k}) \leq G_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k} \) is the set

\[ G_n(\mathbb{R}^{n+k}) \]

of pairs (\( n \)-plane in \( \mathbb{R}^{n+k} \)), vector in said \( n \)-plane).
It comes with a vector space structure

\[ e_1(x, x_1) + e_2(x, x_2) = (x, e_1x_1 + e_2x_2) \]

Local triviality of this bundle

Recall that \( G_n(\mathbb{R}^{n+k}) \) is topologized as a quotient space of \( O(n) \times V_n(\mathbb{R}^{n+k}) \leq \text{Mat}_{n+k}(\mathbb{R}) \), so it has the structure of an

\( (n+k)n - n(n) = (nk) \)-dimensional manifold. We start by constructing a nbhd of any point in \( G_n(\mathbb{R}^{n+k}) \) homeomorphic to \( \mathbb{R}^{nk} \). Let \( X_0 \) be an \( n \)-plane, and \( \mathbb{R}^{nk} = X_0 \oplus X_0^\perp \). Let \( U \subseteq G_n(\mathbb{R}^{n+k}) \) be the set of all \( n \)-plants \( Y \) st orthogonal projection along

\[ \pi: X_0 \oplus X_0^\perp \to X_0 \] takes \( Y \) to \( X_0 \). (That is, \( Y \cap X_0 = \emptyset \).) This is an open condition. Indeed, each \( Y \in U \) is exactly the graph of some

\[ T(Y): X_0 \to X_0^\perp \] so \( T: \text{Hom}(X_0, X_0^\perp) \cong \mathbb{R}^{nk} \), can check this is a homeomorphism.

Now, let \( h: U \times X_0 \to \mathbb{S}^1 \) where \( y \) is the preimage of \( x \)

\[ (y, x) \mapsto (y, y) \]

along the projection \( \pi \) applied to \( Y \).

Then \( h(y, x) = (y, x + T(y)x) \) and \( h^{-1}(y, y) = (y, x) \) is continuous.
Why is this bundle interesting?

Example. Say that $M \subseteq \mathbb{R}^{n+k}$. We can consider the Gauss map $\tilde{g}: M \to G_n(\mathbb{R}^{n+k})$

$$x \mapsto \text{T}_x$$

$$\tilde{g}: (x, v) \mapsto (\text{T}_x, v)$$

$$G_n(\mathbb{R}^{n+k}) \xrightarrow{\tilde{g}}$$

$$\text{T}_M = \tilde{g}^*(\gamma^n(\mathbb{R}^{n+k}))$$

This generalizes

**Prop.** For any $n$-dim bundle $E \to \Theta$ over a compact base $\Theta$ there is a map $\Theta \to G_n(\mathbb{R}^{n+k})$ such that

$$E \cong F^k(\gamma^n(\mathbb{R}^{n+k})) \to \gamma^n(\mathbb{R}^{n+k})$$

(Here $k > 0$)
Proof: Pick $U_1, \ldots, U_r$ covering $\mathcal{B}$ so that $E|_{U_r}$ is trivial. (Finitely many $b/c$ compact). Since $\mathcal{B}$ is normal, $\exists$ open sets $X_1, \ldots, X_r$ covering $\mathcal{B}$ with $\bar{X}_i \subseteq U_i$. Similarly we have $W_1, \ldots, W_r$ w/ $\bar{W}_i \subseteq X_i$. Let $\mathcal{H}: \mathcal{B} \to \mathbb{R}$ be 1 on $\bar{W}_i$ and 0 outside $U_i$. Pick a trivialization $\tilde{\mathcal{H}}_i: \mathcal{B}(U_i) \to \mathbb{R}^n$ which is linear on each fibre.

Define $h_i: E \to \mathbb{R}^n$ by

$$h_i(e) = \begin{cases} 0 & \text{if } p(e) \notin U_i \\ \mathcal{H}_i(p(e)) \tilde{h}_i(e) & \text{for } p(e) \in U_i \end{cases}$$

This is $C^0$ and linear on each fibre. Then we can consider $\mathcal{F}: E \to \mathbb{R}^n \oplus \mathbb{R}^n \oplus \cdots \oplus \mathbb{R}^n$. So an $e \mapsto (h_1(e), \ldots, h_n(e))$.

The isomorphism class of vector bundles is, up to homotopy, a map

$$E = F^* \mathcal{G}(\mathbb{R}^{n+k}) \to \mathcal{G}(\mathbb{R}^{n+k})$$