Lecture 20

Intro to Vector Bundles

Recall We previously had defined complex and real K-theory

Thm Bott Periodicity $\begin{cases} U \simeq \Omega^2 U \\ O \simeq \Omega^8 O \\ Sp \simeq \Omega^8 Sp \end{cases}$ $\begin{cases} O \simeq \Omega^4 Sp \\ Sp \simeq \Omega^4 O \end{cases}$

(Equivalently: Recall $\Omega BU \simeq U$. Bott periodicity can be phrased as $\Omega^2 BU \simeq \mathbb{Z} \times BU$.)

K-theory $\hat{K}^{-i}(X) = \langle X, \Omega^{i-1} U \rangle$ i.e. $\hat{K}^i(X) = \begin{cases} \langle X, BU \rangle & i \text{ even} \\ \langle X, U \rangle & i \text{ odd} \end{cases}$

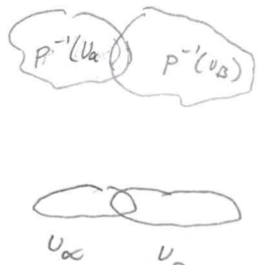
$\widetilde{KO}^i = \langle X, \Omega^{i-1} O \rangle$ i.e. $\tilde{K}^i(X) = \begin{cases} \langle X, BO \rangle & i = 0 \\ \langle X, O \rangle & i = 1 \\ \vdots \text{ etc} \end{cases}$

Ok, so what are these theories?

Defn A vector bundle ξ is a fibre bundle whose fibres are copies of some vector space V and whose transition maps are linear.

V is typically $\mathbb{R}^n$ or $\mathbb{C}^n$

$$\xi \begin{cases} F \longrightarrow E \\ \quad\quad \downarrow p \\ \quad\quad B \end{cases}$$



$$p^{-1}(U_\alpha) \simeq U_\alpha \times \mathbb{R}^n \xrightarrow{\varphi_{\alpha\beta}} U_\beta \times \mathbb{R}^n \simeq p^{-1}(U_\beta)$$

$\varphi_{\alpha\beta}(b, -)$ is linear

Sometimes $E = E(\xi)$

Two vector bundles are said to be isomorphic if there is a homeomorphism between them which is fibre-preserving and linear.

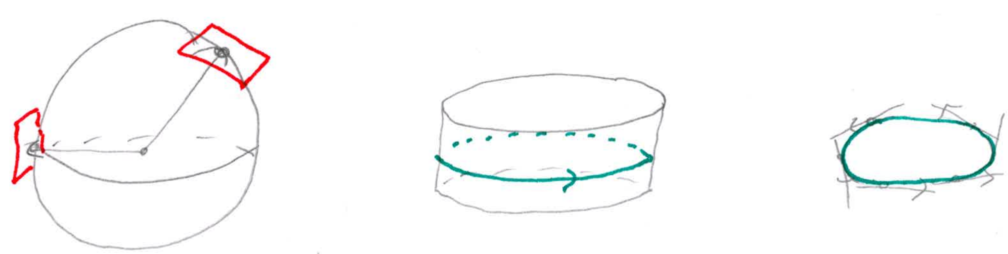
$$\begin{tikzcd} E_1 \arrow[rr, "h"] \arrow[dr, "p_1"'] & \arrow[d, phantom, "G"] & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

- If $\dim(V) = n$, ξ is said to be n-dimensional
- $p^{-1}(b) = F_b$ is the fibre of ξ over b.

Examples

① $\begin{matrix} B \times V \\ \downarrow \\ B \end{matrix}$ is the <u>trivial bundle</u>.

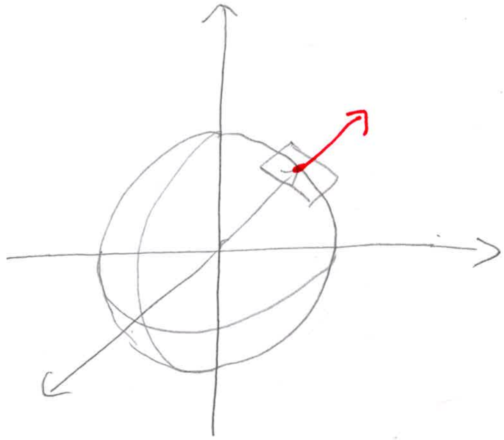
② TM, for M a smooth manifold.



$TS^n_x = \{ v \in \mathbb{R}^n : v \perp x \}$

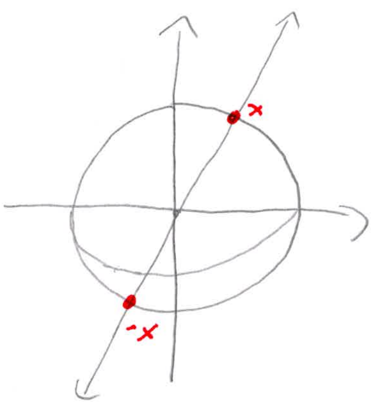
If TM is trivial, we say M is parallelizable.

3) The normal bundle of $M \subseteq \mathbb{R}^n$ smooth, or more generally $M \subseteq N$.



Pick a Riemannian metric and look at the set of vectors orthogonal to $TM_x \subseteq T\mathbb{R}^n_x$, or $TM_x \subseteq TN_x$.

4) The canonical vector bundle over $\mathbb{RP}^n$



$$\{(\{\pm x\}, v) : v = cx\} = \gamma^1_n$$

Local Triviality: Over any $U \subseteq S^n$ not containing a pair of antipodal points, this is just a product.

<u>Exercise</u> What is the canonical line bundle over $\mathbb{RP}^1$?

<u>Defn</u> A <u>section</u> of a vector bundle is a map
$$\begin{tikzcd} E \arrow[d, "p"] \\ B \arrow[u, bend left, "s"] \end{tikzcd}$$
such that $p \circ s$ is the identity. The section is said to be nowhere zero if that is the case. A section of the tangent bundle to a smooth manifold is a <u>vector field</u>.

We see that trivial bundles certainly have n nowhere zero sections, where $n = \dim(V)$.

__Claim__ The canonical line bundle has no nowhere zero section (and is therefore nontrivial).

__Proof__ Suppose we have a section

$$\begin{tikzcd} E(\gamma^1_n) \arrow[d] \\ \mathbb{R}P^n \arrow[u, bend right, "s"'] \end{tikzcd} = (\{\pm x\}, t(x)x).$$

$\{\pm x\}$

We see that $t(x) = -t(-x) \Rightarrow t(x_0) = 0$ for some x_0.

__More generally__

__Defn__ Sections $s_1, \ldots, s_n$ are nowhere dependent if, for each $b \in B$, $s_1(b), \ldots, s_n(b)$ are linearly independent in $p^{-1}(b)$.

__Propn__ $\begin{matrix}\xi \\ \downarrow p \\ B\end{matrix}$ is trivial $\Leftrightarrow$ ξ has n nowhere dependent sections.

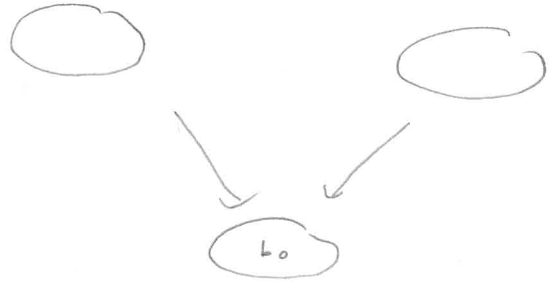
__Comes from__

__Lemma__ Let ξ_1, ξ_2 be vector bundles over B and $E_1 \xrightarrow{f} E_2$ be a cts map which is an isomorphism on each $p_1^{-1}(b) \longrightarrow p_2^{-1}(b)$.

Then f is a homeomorphism (and in particular, a vector space isomorphism).

PF For $b \in B$, find some U over which both ξ_1 and ξ_2 are trivialized.

$$p_1^{-1}(U) \simeq U \times \mathbb{R}^n \xrightarrow{h^{-1} \circ F \circ g} U \times \mathbb{R}^n \simeq p_2^{-1}(U)$$



At each $b \in B$, have $h^{-1} \circ F \circ g\,(b, \vec{x}) = (b, \vec{y}) = (b, (\sum_j F_{ij}(b)\, x_j))$.

↑ entries of a matrix

The matrix (F_{ij}) must vary continuously. So there is a ctsly varying inverse F_{ji}, and we get a homeomorphism.

Proof of Proposition It is clear that a trivial bundle has n nowhere dependent sections. Conversely, if $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ has n nowhere dependent sections,

there is a cts map

$$\begin{tikzcd} E \arrow[r] \arrow[d] & B \times \mathbb{R}^n \arrow[dl] \\ B & \end{tikzcd}$$

which is an isomorphism on each fibre.

Constructing new bundles from old

(a) **Pullback**

$$\begin{tikzcd} f^*(E) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ C \arrow[r, "f"] & B \end{tikzcd} \qquad \xi$$

$f^*(E) = \{(c, v) : p(v) = f(c)\}$

Also called $f^*(\xi)$

In particular, inclusion

$$\begin{tikzcd} E|_{B'} \arrow[r] \arrow[d] & E \arrow[d] \\ B' \arrow[r, hook, "i"'] & B \end{tikzcd}$$

Exercise Homotopic maps induce isomorphic pullback bundles.

Example

$$\begin{tikzcd} T(S^5 \setminus \{x_1, \dots, x_n\}) \arrow[r] \arrow[d] & TS^5 \arrow[d] \\ S^5 \setminus \{x_1, \dots, x_n\} \arrow[r, hook] & S^5 \end{tikzcd}$$

(b) **Cartesian Product**

$$\begin{tikzcd} E_1 \arrow[d] & E_2 \arrow[d] & E_1 \times E_2 \arrow[d] \\ B_1 & B_2 & B \times C \end{tikzcd} \qquad \xi_1 \times \xi_2$$

(c) **Whitney Sum** $E_1 \to B$, $E_2 \to B$

$$\begin{tikzcd} E_1 \oplus E_2 \arrow[r] \arrow[d] & E_1 \times E_2 \arrow[d] \\ B \arrow[r, "\Delta"] & B \times B \end{tikzcd} \qquad \xi_1 \oplus \xi_2$$

e.g. $M \subseteq K$

$TM \oplus NM \simeq TK$

(d) <u>Defn</u> Suppose $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ is a vector bundle ξ. We say ξ': a <u>subbundle</u> if we have $\begin{matrix} E' \\ \downarrow \\ B \end{matrix}$ and $E'_b \subseteq E_b$ is a subspace

For all $b \in B$.

<u>Lemma</u> Let $\begin{matrix} \xi_1 \\ E_1 \\ \downarrow \\ B \end{matrix}$, $\begin{matrix} \xi_2 \\ E_2 \\ \downarrow \\ B \end{matrix}$ be subbundles of $\begin{matrix} \xi \\ E \\ \downarrow \\ B \end{matrix}$ such that

each vector space $(F_1)_b \oplus (F_2)_b = E_b$. Then $\xi_1 \oplus \xi_2 \simeq \xi$.

<u>PF</u> $f: E_1 \oplus E_2 \longrightarrow V$ is an isomorphism on each fibre.

$F(b, e_1, e_2) \longrightarrow (b, e_1 + e_2)$

(e) Many more examples proceed in the way you'd expect:

$Hom(\xi_1, \xi_2)$, $\xi_1 \otimes \xi_2$, $Hom(\xi_1, \mathbb{R})$, $\Lambda^k \xi$, etc.