Lecture 2 — Framed Cobordism & Stable Homotopy Groups

Last Time $\pi_i(X) \to \pi_{i+1}(SX) \to \pi_{i+2}(S^2X) \to \cdots$

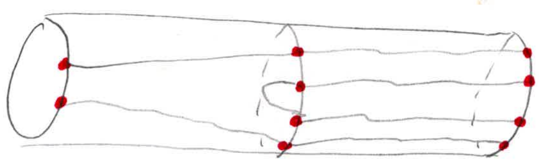
Today · Smooth land, Full smooth approximation.

Let N, N' be cpt n-dim'l submfds of M w/ dim m, $\partial N = \partial N' = \partial M = \emptyset$.

Defn N is cobordant to N' within M if the subset $N \times [0,\varepsilon) \cup N' \times (1-\varepsilon, 1]$ of $M \times [0,1]$ can be extended to a cpt $X \subseteq M \times [0,1]$ w/

$$\partial X = X \cap \big((M \times \{0\}) \cup (M \times \{1\})\big) = \big((N \times \{0\}) \cup (N' \times \{1\})\big).$$

Exercise This is an equivalence relationship.



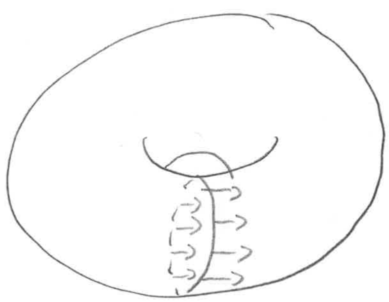
Defn A Framing of $N \subseteq M$ is a smooth function v which assigns to $x \in N$ a basis $v(x) = (v^1(x), \dots, v^{m-n}(x))$ for the space $TN_x^{\perp} \subseteq TM_x$ of normal vectors to N in M at x. Then (N, v) is a framed submanifold.

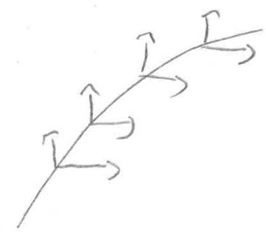
Defn (N, v) and (N', w) are Framed cobordant if $\exists$ a cobordism $X \subseteq M \times [0,1]$ between N and N' and a framing u of X so that

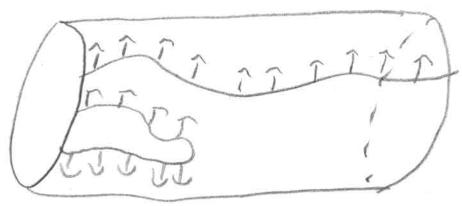
$$u^i(x,t) = (v^i(x), t) \quad \text{For } (x,t) \in N \times [0,\varepsilon)$$

$$u^i(x,t) = (w^i(x), t) \quad \text{For } (x,t) \in N' \times (1-\varepsilon, 1].$$

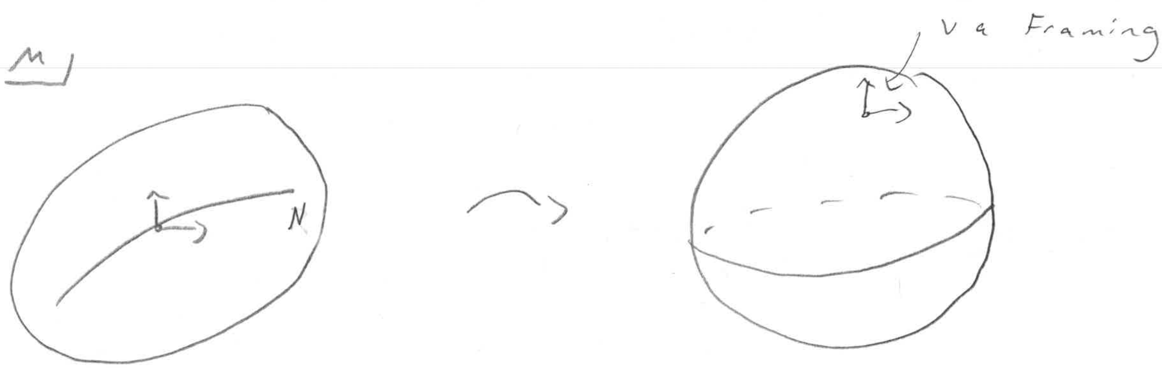
Similarly an equivalence relation.







Let $F: M \to S^p$, $n \geq m$ smooth, y a regular value. Look at $F^{-1}(y)$.



We see $dF_x : T_x M \to T_y S^p$ maps TN to zero, $TN^{\perp} \simeq T_y(S^p)$

Look at $w^i(x) \in TN^{\perp}$ st $dF_x(w^i(x)) = v^i(x)$. Get $w = F^*(v)$ a Framing.

<u>Defn</u> $(F^{-1}(y), F^*v)$ is the <u>Pontrjagin manifold</u> associated to F.

<u>Thm 1</u> If y' is another regular value and v' is another Framing, $(F^{-1}(y'), F^*v')$ is framed cobordant to $(F^{-1}(y), F^*v)$.

<u>Thm 2</u> Two maps $M \to S^p$ are smoothy homotopic $\Leftrightarrow$ their Pontrjagin manifolds are framed cobordant.

<u>Thm 3</u> Any cpt Framed submfd (N, w) of codimension p in M occurs as the Pontrjagin mfd of some $F: M \to S^p$.

Consequence

$$\left\{\begin{array}{c}\text{(smooth) htpy} \\ \text{classes of maps} \\ f: M \to S^p\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Framed } m-p \text{ dim'l} \\ \text{submfds of } M \text{ up} \\ \text{to framed cobordism}\end{array}\right\}$$

Corollary $\pi_{k+p}(S^p) = \{$ Framed cobordism classes of k-dim'l submfds in $S^{k+p}\}$

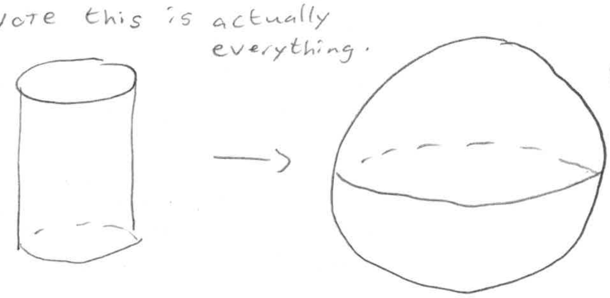
$i = k+p$ $\quad$ $i < 2n-1$

$n = p$ $\quad$ $k+p < 2p-1$

Recall $\pi_{k+p}(S^p) = \pi_k^s$ is stable for $k < p-1$. So this stabilizes and $\pi_k^s = \Omega_k^{Fr}$.

Note this is actually everything.

(k+1)-dim'l cobordism



S^{k+p}

$p-1 > k$

$k+p > 2k+1$

$\Rightarrow k+p \geq 2k+2$

Whitney embedding $M^k \hookrightarrow \mathbb{R}^{2k} \hookrightarrow S^{2k}$ $\longrightarrow$ Stabilizes where Whitney embedding theorem takes effect

$M^{k+1} \hookrightarrow \mathbb{R}^{2k+2} \hookrightarrow S^{2k+2}$

Lemma 1 If v, v' are two different positively oriented bases at y, then $(f^{-1}(y), f^* v)$ is framed cobordant to $(f^{-1}(y), f^* v')$.

Pf Identify the space of positively-oriented framings with $GL^+(p, \mathbb{R})$ and pick a path from v to v'. This path gives the required framing of the cobordism.

Lemma 2 If y is a regular value of F, and z is close to y, then $f^{-1}(z)$ is framed cobordant to $f^{-1}(y)$.

PF Being a regular value is an open condition, so pick a nbhd of y w/ only regular values. If z is in this nbhd, pick a family of rotations $r_t : S^p \to S^p$ w/ $r_1(y) = z$, st

- r_t is identity for $0 \le t \le \varepsilon'$
- $r_t = r_1$ for $1-\varepsilon' < t \le 1$
- $r_t^{-1}(z)$ traces a great circle from z to y.

Let $F : M \times [0,1] \to S^n$. We have z regular for each $r_t \circ F : M \to S^p$

$(x,t) \mapsto r_t F(x)$

$\Rightarrow$ z regular for F, $F^{-1}(z)$ is a framed cobordism between $f^{-1}(z)$ and $(r_1 \circ F)^{-1}(z) = F^{-1}(y)$. $\square$

Lemma 3 $f, g : M \to S^n$ smoothly htpc, y a regular value for both $\Rightarrow$ $f^{-1}(y)$ is framed cobordant to $g^{-1}(y)$.

PF Choose a homotopy F w/ $F(x,t) = f(x)$ $0 \le t < \varepsilon$
$F(x,t) = g(x)$ $1-\varepsilon < t \le 1$

Let z be a regular value for F close enough to y so that $f^{-1}(z)$ is framed cobordant to $f^{-1}(y)$ and $g^{-1}(z)$ is framed cobordant to $g^{-1}(y)$. Then $F^{-1}(z)$ is a framed cobordism between $f^{-1}(z)$ and $g^{-1}(z)$.

Proof of Thm 1 Given two regular values y and z for f, choose rotations $r_t : S^n \to S^n$ st r_0 is identity and $r_1(y) = z$. Then f is homotopic to $r_1 \circ f$, hence $f^{-1}(z)$ is framed cobordant to $(r_1 \circ f)^{-1}(z) = f^{-1}(y)$. □.

Recall (Product Nbhd Thm) Some nbhd of N in M is diffeomorphic to the product $N \times \mathbb{R}^n$. Furthermore the diffeomorphism can be chosen so that $(x, 0)$ corresponds to $x \in N$ and $v(x)$ corresponds to the std basis for $T\mathbb{R}^n$.

Pf Geodesics.

Pf of Thm 3 Let $N \subseteq M$ cpt, boundaryless, framed submanifold. Choose a product representation

$$g : N \times \mathbb{R}^p \to V \subseteq M$$

For a nbhd V of N, as above, and define the projection

$$\pi : V \to \mathbb{R}^p$$

by $\pi(g(x, y)) = y$. 0 is a regular value, and $\pi^{-1}(0)$ is N w/ its framing. Now pick a map φ that maps each $\mathbb{R}^p - \mathring{D}^p$ to a basepoint. Then

$$f(x) = \begin{cases} \varphi \circ \pi(x) & x \in V \\ s_0 & x \notin V \end{cases}$$ is a map $M \to S^p$,

Then 0 is still a regular value, and $\pi^{-1}(0) = N$.