Lecture 19

Last Time $X \circlearrowleft G$ equivariant (co)homology is $H_*^G(X) = H_*(X \times_G EG)$

$$H_G^*(X) = H^*(X \times_G EG)$$

There is a spectral sequence $H^*(X; \mathbb{F}_2) \otimes \mathbb{F}_2[\theta] \Rightarrow H_G^*(X)$ and similarly for homology, which can be obtained as a Serre spectral sequence or from the double complexes:

$$\begin{tikzcd}
C^2 \arrow[r, "1+\tau^\#"] & C^2 \arrow[r, "1+\tau^\#"] & C^2 \\
C^1 \arrow[r, "1+\tau^\#"] \arrow[u, "\partial"] & C^1 \arrow[r, "1+\tau^\#"] \arrow[u, "\partial"] & C^1 \arrow[u, "\partial"] \\
C^0 \arrow[r, "1+\tau^\#"] \arrow[u, "\partial"] & C^0 \arrow[r, "1+\tau^\#"] \arrow[u, "\partial"] & C^0 \arrow[u, "\partial"]
\end{tikzcd}
\qquad \text{or} \qquad
\begin{tikzcd}
C_2 \arrow[d] & C_2 \arrow[l] \arrow[d] & C_2 \arrow[l] \arrow[d] \\
C_1 \arrow[d] & C_1 \arrow[l] \arrow[d] & C_1 \arrow[l] \arrow[d] \\
C_0 & C_0 \arrow[l] & C_0 \arrow[l]
\end{tikzcd}$$

eg E_1 page is now

$$\begin{tikzcd}
H^2(X) \arrow[r] & H^2(X) \arrow[r] & H^2(X) \\
H^1(X) \arrow[r] & H^1(X) \arrow[r] & H^1(X) \\
H^0(X) \arrow[r, "1+\tau^*"] & H^0(X) \arrow[r] & H^0(X)
\end{tikzcd}$$

Why is this a Serre spectral sequence?

$$\begin{tikzcd}
X \arrow[r] & X \times_{\mathbb{Z}/2} S^\infty \arrow[d] \\
& \mathbb{R}P^\infty
\end{tikzcd}$$

The cellular chain complex for S^∞ can be taken to be

$$0 \leftarrow \mathbb{F}_2[\mathbb{Z}/2] \xleftarrow{1+\tau} \mathbb{F}_2[\mathbb{Z}/2] \xleftarrow{1+\tau} \cdots$$

So the singular chain complex for $X \times S^\infty$ can be taken to be

$$0 \leftarrow C_* \underset{\mathbb{F}_2}{\otimes} \mathbb{F}_2[\mathbb{Z}_2] \xleftarrow{1+\tau} C_* \otimes \mathbb{F}_2[\mathbb{Z}_2] \xleftarrow{1+\tau} C_* \otimes \mathbb{F}_2[\mathbb{Z}_2] \leftarrow$$

where the filtration by skeleta of $\mathbb{RP}^\infty$ is exactly by each place in the sequence. We want to dualize and identify elements $x \otimes 1$ and $\tau x \otimes \tau$. The way to do this is to take maps to $\mathbb{F}_2[\mathbb{Z}_2]$

$$0 \to C^* \xrightarrow{1+\tau^*} C^* \xrightarrow{1+\tau^*} C^* \longrightarrow \cdots$$

E^1 page is $H^*(X) \otimes \mathbb{F}_2[\theta]$. E^2 page is called $H(\mathbb{Z}_2, H^*(C_*))$.

<u>Recall</u> There is a second spectral sequence which relates the theory to X^{Fix}. If X is a finite dim'l CW cpx, we have

① There is a spectral sequence $H_*(X) \otimes \mathbb{F}_2[\theta, \theta^{-1}] \Rightarrow H_*(X^{Fix}) \otimes \mathbb{F}_2[\theta, \theta^{-1}]$

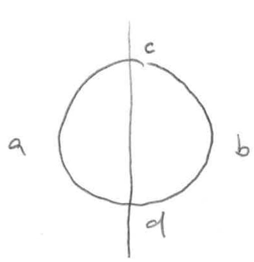
② There is an isomorphism $\theta^{-1} H^*_{\mathbb{Z}_2}(X) \simeq H^*(X^{Fix}) \otimes \mathbb{F}_2[\theta, \theta^{-1}]$.

$\Rightarrow$ <u>Smith Inequality</u> $\dim(H_*(X; \mathbb{F}_2)) \geq \dim(H_*(X^{Fix}; \mathbb{F}_2))$

$\dim(H^*(X; \mathbb{F}_2)) \geq \dim(H^*(X^{Fix}; \mathbb{F}_2))$

<u>Warning</u> Finite dimensionality matters. S^∞ has an involution fixing S^1.

<u>Example</u>



<u>∂</u> $\partial a = \partial b = c + d$

<u>τ</u> $\tau(a) = b \quad \tau(b) = a$

$\tau(c) = c \quad \tau(d) = d$

E^0

$$\begin{tikzcd} \mathbb{F}_2\langle a,b\rangle \arrow[d, "\partial"'] & \mathbb{F}_2\langle a,b\rangle \arrow[l, "1+\tau"'] \arrow[d, "\partial"] & \mathbb{F}_2\langle a,b\rangle \arrow[l] \arrow[d] \\ \mathbb{F}_2\langle c,d\rangle & \mathbb{F}_2\langle c,d\rangle \arrow[l, "1+\tau"'] & \mathbb{F}_2\langle c,d\rangle \arrow[l] \end{tikzcd}$$

Vertical First

E^1

$$\begin{array}{ccc} \mathbb{F}_2\langle [a+b]\rangle & \mathbb{F}_2\langle [a+b]\rangle & \mathbb{F}_2\langle [a+b]\rangle \\ \mathbb{F}_2\langle [c]\rangle & \mathbb{F}_2\langle [c]\rangle & \mathbb{F}_2\langle [c]\rangle \end{array}$$

d^1 $(1+\tau_*)[a+b] = 0$

$(1+\tau_*)[c] = 0$

d^2

$$\begin{tikzcd} 0 & 0 \arrow[l] \arrow[d] \\ & 0 & c \arrow[l, "1+\tau"] \end{tikzcd}$$

Horizontal First

E^1

$$\begin{array}{ccc} \mathbb{F}_2\langle [a]\rangle & & \\ \mathbb{F}_2\langle [c],[d]\rangle & \mathbb{F}_2\langle [c],[d]\rangle & \mathbb{F}_2\langle [c],[d]\rangle \end{array}$$

$\partial[a] = [c+d]$

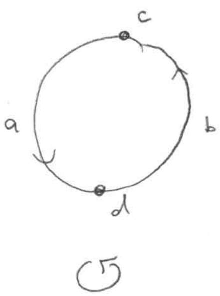
E^2

$$\begin{array}{ccc} \mathbb{F}_2\langle [c]\rangle & \mathbb{F}_2\langle [c],[d]\rangle & \mathbb{F}_2\langle [c],[d]\rangle \end{array}$$

Notice that what we're seeing is

$$\begin{tikzcd} a+b & a \arrow[l] \arrow[d] \\ & c+d \end{tikzcd}$$

Compare to (cohomology for module structure)



$\partial(c) = \partial(d) = a+b$

$\tau(a) = b \quad \tau(b) = a$

$\tau(c) = d \quad \tau(d) = c$

$\underline{E_0}$

$$\begin{tikzcd}
\mathbb{F}_2\langle a,b\rangle \arrow[r, "1+\tau^{\#}"] & \mathbb{F}_2\langle a,b\rangle \arrow[r] & \mathbb{F}_2\langle a,b\rangle \\
\mathbb{F}_2\langle c,d\rangle \arrow[r] \arrow[u, "\partial"'] & \mathbb{F}_2\langle c,d\rangle \arrow[r] \arrow[u] & \mathbb{F}_2\langle c,d\rangle \arrow[u]
\end{tikzcd}$$

<u>Vertical First</u>

E_1/E_2

$$\begin{tikzcd}
\mathbb{F}_2\langle [a]\rangle \arrow[rrd] & \mathbb{F}_2\langle [a]\rangle & \mathbb{F}_2\langle [a]\rangle \\
\mathbb{F}_2\langle [c+d]\rangle & \mathbb{F}_2\langle [c+d]\rangle & \mathbb{F}_2\langle [c+d]\rangle
\end{tikzcd}$$

$(1+\tau^*)[a] = [a+b] = 0$

$(1+\tau^*)[c+d] = 0$

$$\begin{tikzcd}
a \arrow[r, "1+\tau_{\#}"] & a+b \\
 & c \arrow[u, "\partial"] \arrow[r, "1+\tau_{\#}"] & c+d
\end{tikzcd}$$

E_3

$$\begin{tikzcd}
\mathbb{F}_2\langle [c+d]\rangle & \mathbb{F}_2\langle [c+d]\rangle
\end{tikzcd}$$

Horizontal First

E_1

$$\begin{array}{lll}
\mathbb{F}_2\langle [a+b] \rangle & 0 & 0 \\
\mathbb{F}_2\langle [c+d] \rangle & 0 & 0
\end{array}$$

Notice that we're seeing

$$\begin{tikzcd}
a+b & \\
c \arrow[u] \arrow[r] & c+d
\end{tikzcd}$$

We see $e^{-1} H^*_{\mathbb{Z}_2}(X) = 0 = H^*(\phi) \otimes \mathbb{F}_2[e, e^{-1}]$

Notice that the inequality $\dim H^*(X; \mathbb{F}_2) \geq \dim H^*(X^{Fix}; \mathbb{F}_2)$ does not split along gradings (e.g. S^1 vs. S^0). Most accurately,

$$\sum_{i=1}^{\ell} \dim H^i(X; \mathbb{F}_2) \geq \sum_{i=1}^{\ell} \dim H^i(X^{Fix}; \mathbb{F}_2) \quad \forall\, \ell.$$

Digression on Differentials in Double Complexes

Suppose you start with

$$\begin{tikzcd}
A_2 \arrow[d] & B_2 \arrow[l] \arrow[d] & C_2 \arrow[l] \arrow[d] \\
A_1 \arrow[d, "d_0"] & B_1 \arrow[l, "d_1"'] \arrow[d] & C_1 \arrow[l] \arrow[d] \\
A_0 & B_0 \arrow[l] & C_0 \arrow[l]
\end{tikzcd}$$

Say you do vertical first. Then you have some $[b]$ on page E^1 such that $d_0(b) = 0$. But maybe this isn't really a cycle, i.e. maybe we have $a \xleftarrow{d_1} b$. Then $d_0(a)$ is a cycle (the differentials commute)

Then $[a]$ is on page E^1, but it shouldn't be (in the total complex, $a = \partial b$ is a boundary. So do $(1+\tau_*)$ to cancel b and a.

Similarly, if $[c]$ has survived to E^2_1 then $d_1(c) = 0$ and $[d_2(c)] = 0$ on E^1_1, that is $d_1(d_2(c)) = 0$ and furthermore there is some b' such that $d_1(b') = d_2(c)$. Let $a = d_2(b')$. We see that $[c]$ isn't really a cycle in the homology of the whole complex $(\partial c) = d_2(c) \neq 0)$. Furthermore, $d_2(a) = 0$ (by d_2 is a differential) and furthermore $d_1 a = 0$ by

$$\begin{tikzcd} a \arrow[d, dashed] & b' \arrow[l, "d_2"'] \arrow[d, "d_1"] & \\ {} & d_2(c) \arrow[l, dashed] & c \arrow[l, "d_2"'] \end{tikzcd}$$

$$0 = \partial^2(b') = \partial a + \partial(d_2 c) \Rightarrow d_1 a + d_2^2 c = 0 \Rightarrow d_1 a = 0.$$

So $[a]$ is an element on E^2 and shouldn't survive, since actually $a = \partial(b'+c)$ is a boundary. We now cancel $[c]$ with $[a]$. We keep doing this.

Example Suppose $0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$ is an exact sequence of chain complexes.

E_0

$$\begin{tikzcd} 0 & C_3 \arrow[l] \arrow[d] & B_3 \arrow[l] \arrow[d] & A_3 \arrow[l] \arrow[d] & 0 \arrow[l] \\ 0 & C_2 \arrow[l] \arrow[d] & B_2 \arrow[l] \arrow[d] & A_2 \arrow[l] \arrow[d] & 0 \arrow[l] \\ 0 & C_1 \arrow[l] \arrow[d] & B_1 \arrow[l] \arrow[d] & A_1 \arrow[l] \arrow[d] & 0 \arrow[l] \\ 0 & C_0 \arrow[l] & B_0 \arrow[l] & A_0 \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

Horizontal First: The homology of the total complex is zero.

Vertical First

E_1

$$\begin{tikzcd}
H_2(C) & H_2(B) \arrow[l, "g_*^2"'] & H_2(A) \arrow[l, "f_*^2"'] \\
H_1(C) & H_1(B) \arrow[l, "g_*^1"'] & H_1(A) \arrow[l, "f_*^1"'] \\
H_0(C) & H_0(B) \arrow[l, "g_*^0"'] & H_0(A) \arrow[l, "f_*^0"']
\end{tikzcd}$$

f_* is still surjective

E_2

$$\begin{tikzcd}
H_2(C)/im(g_*) & 0 & ker(f_*^2) \\
H_1(C)/im(g_*) & 0 & ker(f_*^1) \arrow[ull, "\sim"] \\
H_0(C)/im(g_*) & 0 & ker(f_*^0) \arrow[ull, "\sim"]
\end{tikzcd}$$

There is an isomorphism

$$ker(f_*^i) \longrightarrow H_i(C)/im(g_*^i)$$

It has some inverse

$$H_i(C)/im(g_*^i) \longrightarrow ker(f_*^{i-1}).$$

So there is a map

$$H_i(C) \xrightarrow{\delta} H_{i-1}(A) \quad w/$$

kernel $im(g_*^i)$ and image $ker(f_*^{i-1})$

$$\longrightarrow H_i(A) \xrightarrow{f_*} H_i(B) \xrightarrow{g_*} H_i(C) \xrightarrow{\delta} H_{i-1}(A) \xrightarrow{f_*} \cdots$$