Say we have a topological space $X$ with an action of a Lie group $G$.

$G = \mathbb{Z}_2$

We can try to compare $H^*(X)$ and $H^*(X^{\text{fix}})$.

Classically, for $\mathbb{Z}_2$ one can look at some long exact sequence involving the quotient, but there is a better way to think about the situation.

- $X/G$ probably loses a bunch of structure because the action of $G$ need not be free.

- Replace with the Borel construction $X \times E G = X \times E G / G \sim (xg, y) \sim (gx, g^*y)$. This is the homotopy quotient of the action.

The equivariant cohomology is $H^*_G(X) = H^*(X \times E G)$. If $G$ finite, typically $H^*_G$ w\ coefficients in $G$.

**Examples**

1. $G = \mathbb{R}$, $H^*_G(\text{pt}) = H^*_G(\mathbb{R}G)$

2. $G$ acts freely $\implies X \times^G E G = X / G \implies H^*_G(X) = H^*(X / G)$.

**Exercise** Our construction also satisfies: If $F : X \longrightarrow Y$ is $G$-equivariant and a homotopy equivalence, then $H^*_G(X) \simeq H^*_G(Y)$. [Note the inverse does not have to be equivariant!]

\[ \text{Reflection} \]
Characterize the theory

\[ x \times E \xrightarrow{\sim} x \]

\( G \) acts freely

\[ \Rightarrow \loom{H^*_G(x)} \cong H^*_G(x \times E) = H^*_G(x \times E_G) \]

Furthermore, have a fibration \( x \xrightarrow{\sim} x \times E_G \)

\[ \downarrow \]

\( E_G \)

1. We see that \( H^*_G(x) \) is a module over \( H^*(E_G) \) via

\[ H^*(x \times E_G) \xleftarrow{\sim} H^*(E_G) \]

and subsequent cup product

2. We also, in principle, have a Serre sequence, perhaps with some coefficient issues.

Specialize to \( \mathbb{Z}_2 \)

- \( G = \mathbb{Z}_2 \), \( E_G = \mathbb{R} \mathbb{P}^\infty \), \( H^*(E_G; \mathbb{F}_2) = \mathbb{F}_2[\Theta] \)

- Serre spectral sequence: \( H^*(x; \mathbb{F}_2) \otimes \mathbb{F}_2[\Theta] \Rightarrow H^*_G(x) \not{E^1} \) page

- All copies of \( H^*(E) = H^*(x; \mathbb{F}_2) \) come identified because the total space is the quotient of a product.
There is also a more tractable way to get this spectral sequence.

- \( X \otimes - \rightarrow C^* \otimes \mathbb{Z}^* \), \((\otimes)^2 = \text{Id}\). We can think of \( C^* \) as an \( \mathbb{F}_2[\mathbb{Z}^*] \)-module.

- Notice that \((1 + 2^*)^2 = 1 + 2^* + (2^*)^2 = 1 + 1 = 0\). This is a new differential.

- When we have two commuting differentials on a complex, we can build a double complex.

\[
\begin{array}{cccc}
C^2 & \rightarrow & C^1 & \rightarrow & C^0 \\
\uparrow \phi & & \uparrow \psi & & \uparrow \psi \\
C^1 & \rightarrow & C^0 & \rightarrow & C^0 \\
\uparrow \phi & & \uparrow \psi & & \uparrow \psi \\
C^0 & \rightarrow & C^0 & \rightarrow & C^0
\end{array}
\]

\( \text{Filtration is by rows (or alternately columns)} \)

**Exercise** This is the same spectral sequence.

Similarly we get a spectral sequence for homology.

**Exercise** We have \( H^\bullet(X; \mathbb{F}_2) = \text{Ext}_{\mathbb{F}_2[\mathbb{Z}^*]}(C^\bullet, \mathbb{F}_2) \), a trivial \( \mathbb{F}_2[\mathbb{Z}^*] \)-module.

**Localization** We claim this has something to do with \( X^{\text{Fix}} \).

Why would you think this? Let's use homology for sanity's sake.

**New hypothesis:** Assume \( X \) is finite-dimensional as a CW-cpx.
Warning: we just used \(\sigma(D^n) \otimes_{\text{Fix}} \Rightarrow 2\sigma(D^{n-1}) \otimes_{\text{Fix}}\), or 
\(C_x(\text{Fix})\) is a legitimate subcomplex of \(C_*(x)\). This is not 
true of general chain complexes \(C_\Delta C^\bullet\).

Localization becomes a harder problem.

Conclusion: a spectral sequence \(H_*(x; \mathbb{F}_2) \otimes \mathbb{F}_2[\theta, \theta^{-1}] \Rightarrow H_*(x^{\text{Fix}}; \mathbb{F}_2)\)

and a corresponding rank inequality

\[\text{rk} (H_*(x; \mathbb{F}_2)) \geq \text{rk} (H_*(x; \mathbb{F}_2))\]

For cohomology, we can also say

\[\theta^{-1} H^*_2(x) \triangleleft H^*(x^{\text{Fix}}) \otimes \mathbb{F}_2[\theta, \theta^{-1}]\]

How did we use the hypothesis? In saying that anything 
coming from the \(A_i\) was definitely \(e\)-torsion.

\[\xymatrix{ \mathbb{F}_2 \ar[d] & \mathbb{Z} \ar[l] \ar[d] \\
\mathbb{F}_2 \ar[r] & \mathbb{Z} \ar[l]}
\]

Otherwise we could have 

some bad situation

where

Not true generally; so has an involution fixing \(S^0\).
The complex comes w/ two spectral sequences. What we've just done is vertical differentials first. What happens if we do horizontal first?

The kernel of $(1+\sigma)$ contains two kinds of generators

\[ \sigma : A^n \to X \text{ such that } \sigma(A^n) \subseteq X_{\text{Fix}} \]

(i.e., such that $\sigma_+ \sigma = \sigma$).

Sums $\sigma + x_\sigma$ where the image of $\sigma$ is not entirely in $X_{\text{Fix}}$.

The second set is the image of $(1+\sigma_x)$. So we can identify the $E'$ page of this spectral sequence w/ $C_* (x_{\text{Fix}}, \mathbb{F}_2)$ in every column except possibly the first. Moreover, the differential is still the ordinary singular differential.

Now, any $[\sigma]$ remaining on this page has $\partial \sigma = (1+\sigma) \sigma = 0$.

So the spectral sequence collapses here. So $H_*^{Z_c}(X) \longrightarrow H_*(x_{\text{Fix}}) \otimes \mathbb{F}_2 [\sigma]$.

Moreover, if we tensor the original complex w/ $\mathbb{F}_2$ we can lose the extra terms.
Example

\[ \varepsilon \alpha \varepsilon \beta = \varepsilon \gamma \varepsilon \delta = \varepsilon \gamma + \varepsilon \delta \]

\[ \tau \alpha = \varepsilon \beta \quad \tau \beta = \varepsilon \gamma \quad \tau \gamma = \varepsilon \delta \quad \tau \delta = \varepsilon \gamma \]

\[ \begin{align*}
E^0 & \to E^1 \\
F_\alpha < a, b > & \sim E F_\beta < a, b > \sim F_\gamma < a, b > \\
\downarrow & \downarrow \\
F_\alpha < c, d > & \sim E F_\beta < c, d > \sim F_\gamma < c, d >
\end{align*} \]

Vertical First

\[ \begin{align*}
E'^0 & \to E'^1 \\
F_\alpha < [a+b] > & \sim E F_\beta < [a+b] > \sim F_\gamma < [a+b] > \\
F_\alpha < [c] > & \sim E F_\beta < [c] > \sim F_\gamma < [c] > \sim F_\beta < [c] >
\end{align*} \]

\[ \begin{align*}
\alpha' & (\varepsilon \beta) \varepsilon [a+b] = 0 \\
(\varepsilon \beta) \varepsilon [c] = 0
\end{align*} \]

So we collapse here.

Horizontal First

\[ \begin{align*}
E'^0 & \to E'^1 \\
F_\alpha < [a+b] > & \sim E F_\beta < [a+b] > \\
F_\alpha < [c] > & \sim E F_\beta < [c] > \sim F_\gamma < [c] > \sim F_\beta < [c] >
\end{align*} \]

\[ \varepsilon [a+b] = \varepsilon [c+\varepsilon \gamma] \]

\[ \varepsilon [a+b] = \varepsilon [c+\varepsilon \gamma] \]
Notice we're seeing \( a + b \leq a \)

Observe that the dimension inequality does not split along gradings.

More accurately,

\[
\sum_{i=0}^{\infty} r_k (H_i (X; \mathbb{F}_2)) \leq \sum_{i=0}^{\infty} r_k (H_i (\tilde{X}; \mathbb{F}_2))
\]