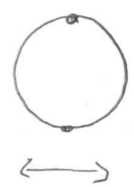
Introduction to Equivariant Cohomology

Say we have a topological space w/ an action of a Lie group G.

eg) $G = \mathbb{Z}_2$



Reflection

We can try to compare $H^*(X)$ and $H^*(X^{Fix})$.

Classically, for $\mathbb{Z}_p$ one can look at some long exact sequence involving the quotient. But there is a better way to think about the situation.

- X/G probably loses a bunch of structure because the action of G need not be free
- Replace w/ the Borel construction $X \times_G EG = X \times EG / (x,y) \sim (gx, gy)$ } Diagonal action

This is the homotopy quotient of the action.

The equivariant cohomology is $H^*_G(X) = H^*(X \times_G EG)$.] If G finite, typically w/ coefficients in G

Examples

① $\cdot \circlearrowleft G$ $\quad H^*_G(pt) = H^*(BG)$

② G acts freely $\rightsquigarrow X \times_G EG = X/G \rightsquigarrow H^*_G(X) = H^*(X/G)$. (A)

Exercise Our construction also satisfies (B): If $f: X \longrightarrow Y$ is G-equivariant and a homotopy equivalence, then $H^*_G(X) \simeq H^*_G(Y)$. [Note the inverse does not have to be equivariant.]

A + B characterize the theory

$$X \times EG \xrightarrow{\sim} X$$

G acts freely

$$\Rightarrow H^*_G(X) \simeq H^*_G(X \times EG) = H^*(X \times_G EG)$$

Furthermore, have a fibration

$$\begin{array}{c} X \hookrightarrow X \times_G EG \\ \downarrow \\ BG \end{array}$$

1 We see that $H^*_G(X)$ is a module over $H^*(BG)$ via

$$H^*(X \times_G EG) \longleftarrow H^*(BG)$$ and subsequent cup product

2 We also, in principle, have a Serre sequence, perhaps w/ same coefficients issues.

Specialize to $\mathbb{Z}_2$

- $G = \mathbb{Z}_2$, $BG = \mathbb{RP}^\infty$, $H^*(BG; \mathbb{F}_2) = \mathbb{F}_2[\theta]$
- Serre spectral sequence: $\underbrace{H^*(X; \mathbb{F}_2) \otimes \mathbb{F}_2[\theta]}_{E' \text{ page}} \Rightarrow H^*_G(X)$
- All copies of $H^*(F) = H^*(X; \mathbb{F}_2)$ come identified because the total space is the quotient of a product.

$$\begin{array}{lll}
H^2(X;\mathbb{F}_2) & H^2(X;\mathbb{F}_2) & H^2(X;\mathbb{F}_2) \\
H^1(X;\mathbb{F}_2) & H^1(X;\mathbb{F}_2) & H^1(X;\mathbb{F}_2) \\
H^0(X;\mathbb{F}_2) & H^0(X;\mathbb{F}_2) & H^0(X;\mathbb{F}_2)
\end{array}$$

There is also a more tractable way to get this spectral sequence.

- $X \circlearrowleft \tau \rightsquigarrow C^* \circlearrowleft \tau^*$, $(\tau^*)^2 = Id$. We can think of C_* as an $\mathbb{F}_2[\mathbb{Z}_2]$-module.
- Notice that $(1+\tau^*)^2 = 1 + \overset{0}{\cancel{2\tau^*}} + (\tau^*)^2 = 1+1 = 0$. This is a new differential
- When we have two commuting differentials on a complex, we can build a double complex.

$$\begin{array}{ccccc}
C^2 & \xrightarrow{1+\tau^*} & C^2 & \xrightarrow{1+\tau^*} & C^2 \\
\uparrow \partial & & \uparrow \partial & & \uparrow \partial \\
C^1 & \xrightarrow{1+\tau^*} & C^1 & \xrightarrow{1+\tau^*} & C^1 & \cdots \\
\uparrow \partial & & \uparrow \partial & & \uparrow \partial \\
C^0 & \xrightarrow{1+\tau^*} & C^0 & \xrightarrow{1+\tau^*} & C^0
\end{array}$$

Filtration is by rows (or alternately columns)

Exercise This is the same spectral sequence.

Similarly we get a spectral sequence for homology.

Exercise We have $H^*_{\mathbb{Z}_2}(X;\mathbb{F}_2) \simeq \mathrm{Ext}_{\mathbb{F}_2[\mathbb{Z}_2]}(C_*, \underset{\text{trivial } \mathbb{F}_2[\mathbb{Z}_2]\text{-module.}}{\mathbb{F}_2})$

Localization We claim this has something to do w/ X^{Fix}.

Why would you think this? Let's use homology for sanity's sake.

New hypothesis: Assume X is finite-dimensional as a CW-cpx.

<u>Warning</u> we just used $\partial\sigma(\Delta^n) \in X^{Fix} \Rightarrow \partial\sigma(\Delta^{n-1}) \in X^{Fix}$, or

$C_*(X^{Fix})$ is a legitimate subcomplex of $C_*(X)$. This is not

true of general chain complexes $C_* \circlearrowleft \tau_*$.

$$\begin{tikzcd} & a = \tau(a) \arrow[dl] \arrow[dr] & \\ b & & \tau(b) \end{tikzcd}$$

Localization becomes a harder problem

<u>Conclusion</u> $\exists$ a spectral sequence $H_*(X;\mathbb{F}_2) \otimes \mathbb{F}_2[\theta,\theta^{-1}] \Rightarrow H_*(X^{Fix};\mathbb{F}_2) \otimes \mathbb{F}_2[\theta,\theta^{-1}]$

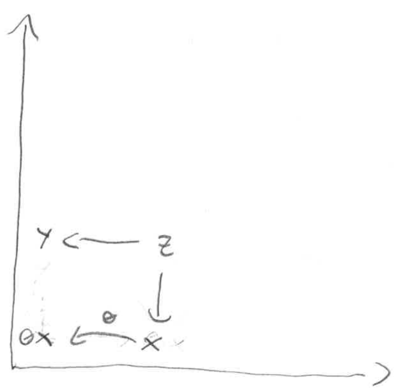
and a corresponding rank inequality

$$rk(H_*(X;\mathbb{F}_2)) \geq rk(H_*(X;\mathbb{F}_2))$$

For cohomology, we can also say

$$\theta^{-1} H^*_{\mathbb{Z}/2}(X) \cong H^*(X^{Fix}) \otimes \mathbb{F}_2[\theta,\theta^{-1}]$$

How did we use the hypothesis? In saying that anything coming from the A_i was definitely θ-torsion.



Otherwise we could have some bad situation where

<u>Not true generally</u>: S^∞ has an involution fixing S^0.

The complex comes w/ two spectral sequences. What we've just done is vertical differentials first.

What happens if we do horizontal first?

E^1

$$\begin{tikzcd}
\arrow[d] & \arrow[d] & \arrow[d] & \\
C_2 \arrow[d] & C_2 \arrow[l] \arrow[d] & C_2 \arrow[l] \arrow[d] & \arrow[l] \\
C_1 \arrow[d] & C_1 \arrow[l] \arrow[d] & C_1 \arrow[l] \arrow[d] & \arrow[l] \\
C_0 & C_0 \arrow[l] & C_0 \arrow[l] & \arrow[l]
\end{tikzcd}$$

The kernel of $(1+\tau)$ contains two kinds of generators

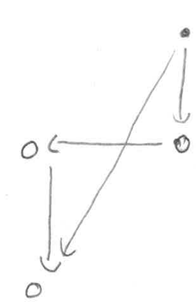
- $\sigma : \Delta^n \to X$ such that $\sigma(\Delta^n) \subseteq X^{Fix}$ (i.e., such that $\tau_* \sigma = \sigma$).
- Sums $\sigma + \tau_* \sigma$ where the image of σ is not entirely in X^{Fix}.

The second set is the image of $(1+\tau_*)$. So we can identify the E^1 page of this spectral sequence w/ $C_*(X^{Fix}; \mathbb{F}_2)$ in every column except possibly the first. Moreover, the differential is still the ordinary singular differential.

E^2

$$\begin{array}{lll}
A_2 \oplus H_2(X^{Fix}) & H_2(X^{Fix}) & H_2(X^{Fix}) \\
A_1 \oplus H_1(X^{Fix}) & H_1(X^{Fix}) & H_1(X^{Fix}) \\
A_0 \oplus H_0(X^{Fix}) & H_0(X^{Fix}) & H_0(X^{Fix})
\end{array}$$

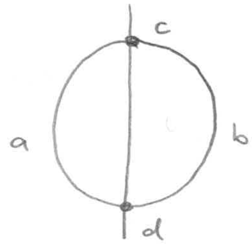
Now, any $[\sigma]$ remaining on this page has $\partial\sigma = (1+\tau)\sigma = 0$



So the spectral sequence collapses here. So $H_*^{\mathbb{Z}_2}(X) \longrightarrow H_*(X^{Fix}) \otimes \mathbb{F}_2[e]$.

Moreover, if we tensor the original complex w/ e we can lose the extra terms.

Example



∂] $\partial a = \partial b = c + d$

τ] $\tau(a) = b \quad \tau(b) = a$

$\tau(c) = c \quad \tau(d) = d$

E^0]

$$
\begin{tikzcd}
\mathbb{F}_2\langle a,b\rangle \arrow[d, "\partial"] & \mathbb{F}_2\langle a,b\rangle \arrow[l, "1+\tau"'] \arrow[d, "\partial"] & \mathbb{F}_2\langle a,b\rangle \arrow[l] \arrow[d, "\partial"] \\
\mathbb{F}_2\langle c,d\rangle & \mathbb{F}_2\langle c,d\rangle \arrow[l, "1+\tau"'] & \mathbb{F}_2\langle c,d\rangle \arrow[l]
\end{tikzcd}
$$

Vertical First

E^1]

$$
\begin{array}{lll}
\mathbb{F}_2\langle [a+b]\rangle & \mathbb{F}_2\langle [a+b]\rangle & \mathbb{F}_2\langle [a+b]\rangle \\
\mathbb{F}_2\langle [c]\rangle & \mathbb{F}_2\langle [c]\rangle & \mathbb{F}_2\langle [c]\rangle
\end{array}
$$

d^1] $(1+\tau)_*[a+b] = 0$

$(1+\tau)_*[c] = 0$

d^2]
$$
\begin{tikzcd}
0 & 0 \arrow[l] \arrow[d] \\
0 & c \arrow[l, "1+\tau"']
\end{tikzcd}
$$

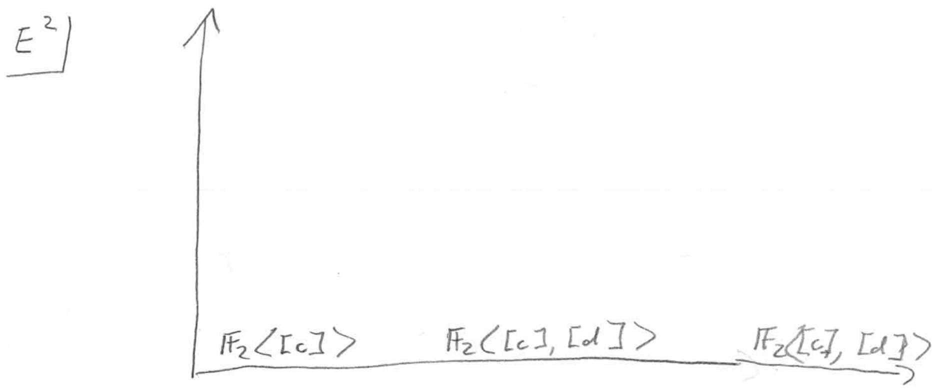
So we collapse here.

Horizontal First

E^1]

$$
\begin{array}{lll}
\mathbb{F}_2\langle [a]\rangle & & \\
\mathbb{F}_2\langle [c],[d]\rangle & \mathbb{F}_2\langle [c],[d]\rangle & \mathbb{F}_2\langle [c],[d]\rangle
\end{array}
$$

$\partial[a] = [c+d]$



Notice we're seeing

$$\begin{tikzcd} a+b & a \arrow[l] \arrow[d] \\ & c+d \end{tikzcd}$$

Observe that the dimension inequality does not split along gradings.

<u>More accurately</u> $\displaystyle\sum_{i=0}^{\ell} rk\,(H_i(X;\mathbb{F}_2)) \geq \sum_{i=0}^{\ell} rk\,(H_i(X^{\mathbb{Z}_2};\mathbb{F}_2))$