Lecture 17

Rational Homotopy Theory

Let $\mathcal{C}$ be one of
$$\begin{cases} \mathcal{FG} & \text{Finitely generated abelian groups} \\ \tau_P & P \text{ a set of primes, abelian groups w/ order}(g) \in P^n \text{ for some } n \ \forall g \in G \\ (\tau = \bigcup_p \tau_p) & \\ \mathcal{F}_P & \text{Finely generated abelian groups in } \tau_P \text{ (necessarily finite)} \end{cases}$$

Thm 1 Let X be an abelian topological space ($\pi_1(X)$ acts trivially on all $\pi_n(X)$). Then $\pi_k(X) \in \mathcal{C} \ \forall k>0 \iff H_k(X) \in \mathcal{C} \ \forall k>0$

"Contractible modulo $\mathcal{C}$"

Example 1 $X = S^n$. We know $H_k(S^n) \subseteq \mathcal{FG} \Rightarrow \pi_k(S^n) \subseteq \mathcal{FG} \ \forall k$

Example 2 X abelian is necessary. Consider $S^1 \vee S^2$. $H_k(S^1 \vee S^2)$ is finitely generated, but $\pi_2(S^1 \vee S^2)$ is infinitely generated.

 } universal cover

Example 3 Similarly $\mathbb{RP}^{2n}$. $H_*(\mathbb{RP}^{2n}) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}/2\mathbb{Z} & k \text{ odd} \\ 0 & k>0 \text{ even} \end{cases}$ but $\pi_{2n}(\mathbb{RP}^{2n}) \simeq \pi_{2n}(S^{2n}) \simeq \mathbb{Z} \notin \tau_2$

More generally Let $\mathcal{C}$ be any class of abelian groups st

1. If $0 \to A \to B \to C \to 0$ is exact, $B \in \mathcal{C} \iff A, C \in \mathcal{C}$
2. $A, B \in \mathcal{C} \Rightarrow A \otimes B \in \mathcal{C}$, $\mathrm{Tor}_1(A, B) \in \mathcal{C}$

Then $\mathcal{C}$ is a Serre class.

<u>Thm 2</u> X abelian, $\pi_k(X) \in \mathcal{C}$ for $1 \le k < n$. Then $H_k(X) \in \mathcal{C}$ for $1 \le k < n$ and moreover $h: \pi_n(X) \longrightarrow H_n(X)$ is an <u>isomorphism modulo $\mathcal{C}$</u>.

Note that, eg, "isomorphism modulo $\tilde{\mathcal{C}}$" means the map becomes an isomorphism after tensoring with $\mathbb{Q}$.

<u>More Vocabulary</u>

A map $F: X \to Y$ of abelian spaces is a <u>quasi-isomorphism modulo $\mathcal{C}$</u> if $F_*: H_*(X) \longrightarrow H_*(Y)$ is an isomorphism mod $\mathcal{C}$. X, Y are homotopy equivalent mod $\mathcal{C}$ if they can be connected by a quasi-isomorphism mod $\mathcal{C}$ (e.g. "<u>rational homotopy equivalence</u>" is mod $\tilde{\mathcal{C}}$)

<u>PF of Thm 1</u>

<u>Lemma</u> $F \to X \to B$ a fibration, all spaces abelian, $\pi_1(B)$ acts trivially on the homology of the fibre. If any two spaces have homology groups in $\mathcal{C}$ for $k \ge 1$, so does the third.

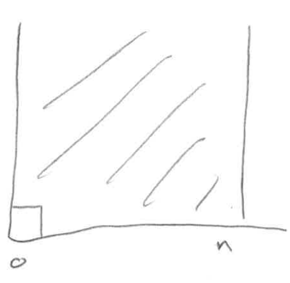
<u>PF</u> <u>Case I</u> F, B have homology groups in $\mathcal{C}$. Have a spectral sequence converging to $H_*(X)$ w/

$$E^2_{p,q} = H_p(B; H_q(F)) \cong (H_p(B) \otimes H_q(F)) \oplus \mathrm{Tor}(H_{p-1}(B), H_q(F))$$

$\in \mathcal{C}$ for $(p,q) \neq (0,0)$

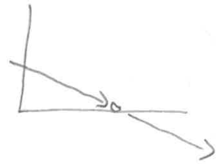
So if $(p,q) \neq (0,0)$, $E^2_{p,q} \in \mathcal{C}$. But then $E^3_{p,q} = \ker(d^2)/\mathrm{im}(d^2) \in \mathcal{C}$ by the ses condition $\rightsquigarrow E^\infty_{p,q} \in \mathcal{C} \rightsquigarrow H_k(X) \in \mathcal{C}$.

<u>Case II</u> X, F have homology groups in $\mathcal{C}$. We know $E^\infty_{p,q} \in \mathcal{C}$ for $(p,q) \neq (0,0)$. Suppose we know $H_k(B) \in \mathcal{C}$ for $1 \le k < n$.



At least the shaded part of the quadrant is in $\mathcal{C}$.

Notice that the part of the spectral sequence known to be in $\mathcal{C}$ has to stay in $\mathcal{C}$ on subsequent pages. Now, by assumption $E^{\infty}_{n,0} \in \mathcal{C}$, but $E^{\infty}_{n,0} = E^{n+1}_{n,0}$, since we must have stabilized at $(n,0)$ by this page



. Ergo, we can now use

$d^r : E^r_{n,0} \longrightarrow E^r_{n-r,r-1}$ to produce an ses

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E^{r+1}_{n,0} & \longrightarrow & E^r_{n,0} & \longrightarrow & \mathrm{Im}(d^r) & \longrightarrow & 0 \\ & & & & & & \rotatebox{90}{$\supseteq$} & & \\ & & & & & & \left.E^r_{n-r,r-1}\right\} \in \mathcal{C} & & \end{array}$$

to say $E^{r+1}_{n,0} \in \mathcal{C} \Leftrightarrow E^r_{n,0}$ is. So $E^{n+1}_{n,0} \in \mathcal{C} \Rightarrow E^2_{n,0} \in \mathcal{C} \Rightarrow$ $H_n(B; H_0(F)) \in \mathcal{C} \Rightarrow H_n(B) \in \mathcal{C}$. $\square$

<u>Case III</u> $H_*(X)$, $H_*(B)$ are in $\mathcal{C}$. Extremely similar (exercise).

<u>Lemma</u> $G \in \mathcal{C} \Rightarrow H_k(K(G,n)) \in \mathcal{C}$ for $k \geq 1$.

<u>Pf</u> Path-loop Fibration

$$\begin{array}{ccc} K(G,n) & \longrightarrow & PK(G,n+1) \\ & & \downarrow \\ & & K(G,n+1) \end{array}$$

$\rightsquigarrow H_*(K(G,n)) \in \mathcal{C}$

$\Leftrightarrow H_*(K(G,n+1)) \in \mathcal{C}$.

Consider $K(G,1)$.

- If $G = \mathbb{Z}$, $H_*(S^1)$ is finitely generated.

- If $G = \mathbb{Z}/n\mathbb{Z}$, $K(\mathbb{Z}/n\mathbb{Z}, 1) = S^\infty/(\mathbb{Z}/n\mathbb{Z})$ an infinite lens space.

$$H_k(K(\mathbb{Z}/n\mathbb{Z},1)) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/n\mathbb{Z} & k \text{ odd} \\ 0 & k \text{ even} \end{cases} \Bigg\} \text{ all finite}$$

- Now let $\mathcal{C} = \mathcal{F}G$. $G \in \mathcal{F}G \Rightarrow G = \prod_i (\mathbb{Z}/n_i\mathbb{Z})$. $K(G,1) = \prod_i K(\mathbb{Z}/n_i\mathbb{Z}, 1)$
$\Rightarrow K(G,1)$ has finitely generated homology.

- $\mathcal{C} = \mathcal{F}_p$ is similar

- $\mathcal{C} = \tau_p$: Canonical $\overset{BG}{\overset{\shortparallel}{K(G,1)}}$ constructed as the simplicial complex $[g_1|\dots|g_k]$ where $g_i \in G$, as we did in class. If H is a subgroup of G, there is a canonical map $K(H,1) \hookrightarrow K(G,1)$. Now for any $[\alpha] \in H_*(K(G,1))$, α maps to finitely many simplices in $K(G,1) \Rightarrow \alpha \in \operatorname{im}(K(H,1) \hookrightarrow K(G,1))$ for H a finitely generated subgroup. So $[\alpha] = \iota_*[\alpha']$ for some $[\alpha'] \in H_*(K(H,1))$. $[\alpha']$ has torsion of order divisible by $p \Rightarrow$ the same is true of $[\alpha]$.

<u>Proof of Thm 1</u>

$(\pi_k \in \mathcal{C} \Rightarrow H_k \in \mathcal{C})$

X is an abelian space, so there is a Postnikov tower

$$\begin{tikzcd} K(\pi_3 X, 3) \arrow[r, hook] & X_3 \arrow[d] \\ K(\pi_2 X, 2) \arrow[r, hook] & X_2 \arrow[d] \\ X \arrow[r] & X_1 = K(\pi_1 X, 1) \end{tikzcd}$$

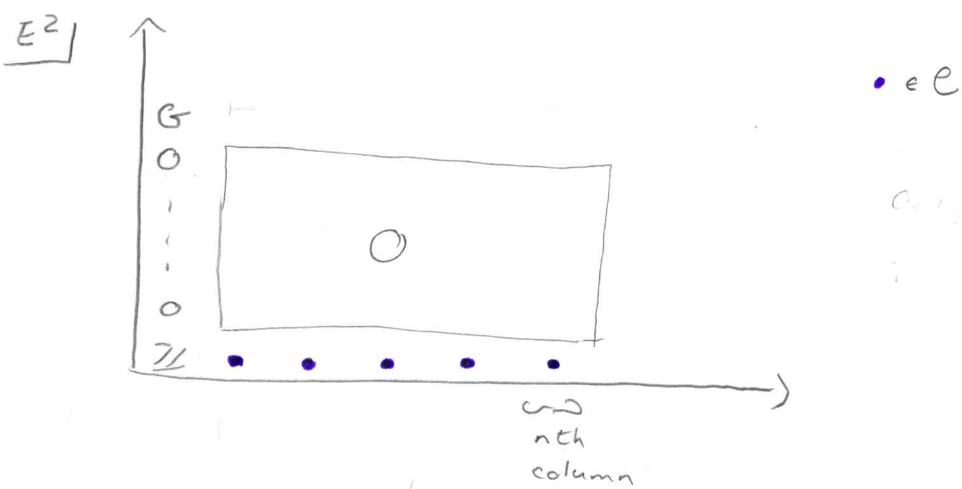
We know $H_n(X_n) \simeq H_n(X)$. We iterate the lemmas over the tower of fibrations to imply $H_k(X_n) \in \mathcal{C}$ for all $X_n \Rightarrow H_k(X) \in \mathcal{C}$ for all k.

($\Leftarrow$) is a consequence of Thm 2, see below.

<u>Pf of theorem 2</u> Say $\pi_k(X) \in \mathcal{C}$ for $k \leq n$. Then $H_k(X_{n-1}) \in \mathcal{C}$ for all k. Now consider the fibration

$$\begin{tikzcd} K(G,n) \arrow[r] & X_n \arrow[d] \\ & X_{n-1} \end{tikzcd} \qquad \text{for } G = \pi_n(X)$$

There is a Serre spectral sequence.



Only interesting differential is $d^n: \underset{\substack{\cup \\ \mathcal{C}}}{H_{n+1}(X_{n-1})} \to \underset{\substack{\wr\vert \\ G}}{H_n(F)} \to E^{\infty}_{0,n} \to 0$

We see that $E^{\infty}_{0,n} \simeq G \mod \mathcal{C}$

Filtration of $H_n(X_n)$:

$$0 \longrightarrow \underset{\substack{\uparrow \\ \text{isomorphism} \\ \text{mod } \mathcal{C}}}{E^{\infty}_{0,n}} \longrightarrow H_n(X_n) \longrightarrow \underset{\substack{\rotatebox{90}{$\in$} \\ \mathcal{C}}}{H_n(X_{n-1})} \longrightarrow 0$$

So we have

$$\begin{tikzcd}
0 \arrow[r] & H_{n+1}(X_{n-1}) \arrow[r] & H_n(F) \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\cong$}"] & H_n(X_n) \arrow[r] \arrow[dd, phantom, "\rotatebox{90}{$\cong$}"] & H_n(X_{n-1}) \arrow[r] & 0 \\
& & G \arrow[d, phantom, "\rotatebox{90}{$\cong$}"] & & & \\
& & \pi_n(X) \arrow[r, "h"] & H_n(X) & &
\end{tikzcd}$$

So the Hurewicz map is an isomorphism modulo $\mathcal{C}$.

Introduction to Equivariant Cohomology

Say we have a topological space w/ an action of a Lie group G.

eg) $G = \mathbb{Z}_2$

Reflection

We can try to compare $H^*(X)$ and $H^*(X^{Fix})$.

Classically, for $\mathbb{Z}_p$ one can look at some long exact sequence involving the quotient. But there is a better way to think about the situation.

- X/G probably loses a bunch of structure because the action of G need not be free
- Replace w/ the Borel construction $X \times_G EG = X \times EG \big/ \underbrace{(x,y) \sim (gx, gy)}$ } Diagonal action

This is the homotopy quotient of the action.

The equivariant cohomology is $H^*_G(X) = H^*(X \times_G EG)$.] If G finite, typically w/ coefficients in G

Examples

① $\cdot \circlearrowleft G$ $\quad H^*_G(pt) = H^*(BG)$

② G acts freely $\rightsquigarrow X \times_G EG = X/G \rightsquigarrow H^*_G(X) = H^*(X/G)$. Ⓐ

Exercise Our construction also satisfies Ⓑ: If $f: X \longrightarrow Y$ is G-equivariant and a homotopy equivalence, then $H^*_G(X) \simeq H^*_G(Y)$. [Note the inverse does not have to be equivariant.]

A + B characterize the theory

$$X \times EG \xrightarrow{\sim} X$$

G acts Freely

$$\Rightarrow H^*_G(X) \simeq H^*_G(X \times EG) = H^*(X \times_G EG)$$

Furthermore, we have maps

$$\begin{array}{c} X \times_G EG \\ \downarrow \\ BG \end{array} \quad \text{and} \quad \begin{array}{ccc} BG & \longrightarrow & X \times_G EG \\ & & \downarrow \\ & & X \end{array}$$

} This one is a Fibration.

① We see that $H^*_G(X)$ is a module over $H^*(BG)$ via

$H^*(X \times_G EG) \longleftarrow H^*(BG)$ and subsequent cup product.

② We also, in principle, have a Serre spectral sequence, possibly w/ some coefficients issues.

Specialize to $\mathbb{Z}_2$

- $G = \mathbb{Z}_2$, $BG = \mathbb{R}P^\infty$, $H^*(BG; \mathbb{Z}_2) = \mathbb{Z}_2[\theta]$
- Serre spectral sequence always converges: $\nexists$ more than one way to identify copies of $H^q(B\mathbb{Z}_2; \mathbb{Z}_2) = \mathbb{Z}_2$ $\forall q \geq 0$.
- So we have a spectral sequence that looks like

$$\begin{array}{|lll} \vdots & \vdots & \vdots \\ H^0(X;\mathbb{Z}_2)\theta & H^1(X;\mathbb{Z}_2)\theta & H^2(X;\mathbb{Z}_2)\theta \\ H^0(X;\mathbb{Z}_2) & H^1(X;\mathbb{Z}_2) & H^2(X;\mathbb{Z}_2) \\ \hline \end{array}$$

$$H^*(X;\mathbb{Z}_2) \otimes \mathbb{Z}_2[\theta] \Rightarrow H^*_G(X).$$

In Fact it's normal to do a reflection and Flip p and q.

Next time: Obtaining this ss in a computable way, and localization