Lecture 16

Recall we have

$$\begin{tikzcd} F \arrow[r, hook] & X \arrow[d] \\ & B \end{tikzcd}$$

B ← to avoid unpleasant technical issue, assume simply ctd

$E_2^{p,q} = H^p(B; H^q(F)) \Rightarrow H^*(X)$

$\exists$ a product $E_n^{p,q} \times E_n^{s,t} \longrightarrow E_n^{p+s,\, q+t}$ st

- On page 2, the product is $(-1)^{qs}$ multiplied by ordinary cup product, or over a Field the ordinary cup product on $H^p(B) \otimes H^q(F)$.
- $d_n(\alpha \cdot \beta) = d_n\alpha \cdot \beta + (-1)^{|\alpha|} \alpha \cdot d_n\beta$, so product on E_n induces product on E_{n+1}
- The product on E_∞ is the product on $gr\, H^*(X)$ induced by the cup product.

Main Idea of the Proof: Spectral sequences are natural wrt Filtered maps.

Propn Given

$$\begin{tikzcd} F \arrow[r, "f'"] \arrow[d] & \bar{F} \arrow[d] \\ X \arrow[r, "\tilde{F}"] \arrow[d] & \bar{X} \arrow[d] \\ B \arrow[r, "f"'] & \bar{B} \end{tikzcd}$$

1. There are maps $f_*^r : E^r_{p,q} \longrightarrow \bar{E}^r_{p,q}$ commuting w/ the differentials d_r, and f_*^{r+1} the map induced on homology by f_*^r.

2. The map $\tilde{f}_* : H_*(X; R) \longrightarrow H_*(\bar{X}; R)$ is filtration-preserving and f_*^∞ is the induced map on associated gradeds

3. $F_*^2 : H_p(B; H_q(F)) \longrightarrow H_p(\bar{B}; H_q(\bar{F}; R))$. & likewise for cohomology

From this

Corollary 1 The Serre spectral sequence from the E^2 page forward does not depend on the cell structure put on B.

Corollary 2 Given

$$\begin{tikzcd} & F \arrow[dl] \arrow[dr] & \\ \bar{X} \arrow[rr, hook] \arrow[d] & & X \arrow[d] \\ \bar{B} \arrow[rr] & & B \end{tikzcd}$$

there is a relative Serre spectral sequence w/ $E^2_{p,q} = H_p(B, \bar{B}; H_q(F;R))$ converging to $H^*(X, \bar{X}; R)$

Corollary 3 The product structure on the cohomological Serre spectral sequence.

Structure Recall the cup product is $H^*(X) \otimes H^*(X) \xrightarrow{\times} H^*(X \times X) \xrightarrow{\Delta^*} H^*(X)$ where $\times$ is the cross product (given on cochains by a straightforward evaluation on simplices) followed by the map Δ^* induced by $X \xrightarrow{\Delta} X \times X$. We look at

$$\begin{tikzcd} F \arrow[r, "\Delta"] \arrow[d] & F \times F \arrow[d] \\ X \arrow[r, "\Delta"] \arrow[d] & X \times X \arrow[d] \\ \bar{B} \arrow[r, "\Delta"] & B \times B \end{tikzcd}$$

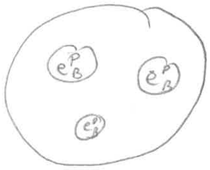
and write out a lengthy but uncomplicated computation for the cross product.

$\leftarrow$ This is why simply connected

Proof of Propn Suffices to check for $B, \bar{B}$ CW complexes. Homotop F to a cellular map, so that $\hat{F}(X_p) \subseteq \bar{X}_p$, i.e., $\hat{F}$ is a map of filtered topological spaces. This implies $\tilde{F}$ induces maps

$A'_{p,q} \xrightarrow{\bar F} \bar A'_{p,q}$, $E'_{p,q} \xrightarrow{F'_*} \bar E'_{p,q}$ commuting w/ the exact couple maps, From which we can induce maps on every subsequent page of the spectral sequence. [In particular, $\bar F(i^r(A)) \subseteq i^r(A)$ for all r, and F^r_* commutes w/ $j^r \circ k^r$ for all r.] It remains to show we have the correct maps on the E^2 page. Let $\Psi : H_{p+q}(X_p, X_{p-1}) \longrightarrow H_p(B^p, B^{p-1}; \mathbb{Z}) \otimes H_q(F; R)$

be the isomorphism we described in class, similarly $\bar\Psi$. Let us assume, if e^p_α denotes a cell in B^p and e^p_β a cell in $\bar B^p$, that $F|_{e^p_\alpha}$ maps disjoint disks homeomorphically to the interiors of cells e^p_β and otherwise is mapped to $\bar B^{p-1}$.



e^p_α

Let $\Phi_\alpha : e^p_\alpha \hookrightarrow B^p$

$\tilde\Phi_\alpha : (\tilde D^p_\alpha, \tilde S^{p-1}_\alpha) \longrightarrow (X_p, X_{p-1})$

We have

$$\begin{tikzcd}
H_{p+q}(\tilde D^p_\alpha, \tilde S^{p-1}_\alpha) \arrow[r] \arrow[d, "\tilde F_*(\hat\Phi_\alpha)_*"] \arrow[drr] & H_{p+q}(\tilde D^p_\alpha, \tilde D^p_\alpha - \mathrm{int}(\bigcup_i \tilde D_i^p)) \arrow[d] & \bigoplus_i H_{p+q}(\tilde D_i^p, \tilde S_i^{p-1}) \arrow[l, "\sim"'] \arrow[d] \\
H_{p+q}(\bar X_p, \bar X_{p-1}) \arrow[r] & H_{p+q}(\bar X_p, \bar X_{p-1} - \tilde e^p_\beta) & H_{p+q}(\tilde D^p_\beta, \tilde S^{p-1}_\beta) \arrow[l, "\sim"']
\end{tikzcd}$$

We now just have to check for a map from the disk to itself

$$\begin{tikzcd}
H_{p+q}(\hat D^p_\alpha, \hat S^{p-1}_\alpha) \arrow[r, "\Psi"] \arrow[d, "f_*"] & H_q(F) \arrow[d, "f'_*"] \\
H_{p+q}(\tilde D^p_\beta, \hat S^{p-1}_\beta) \arrow[r, "\bar\Psi"] & H_q(F)
\end{tikzcd}$$

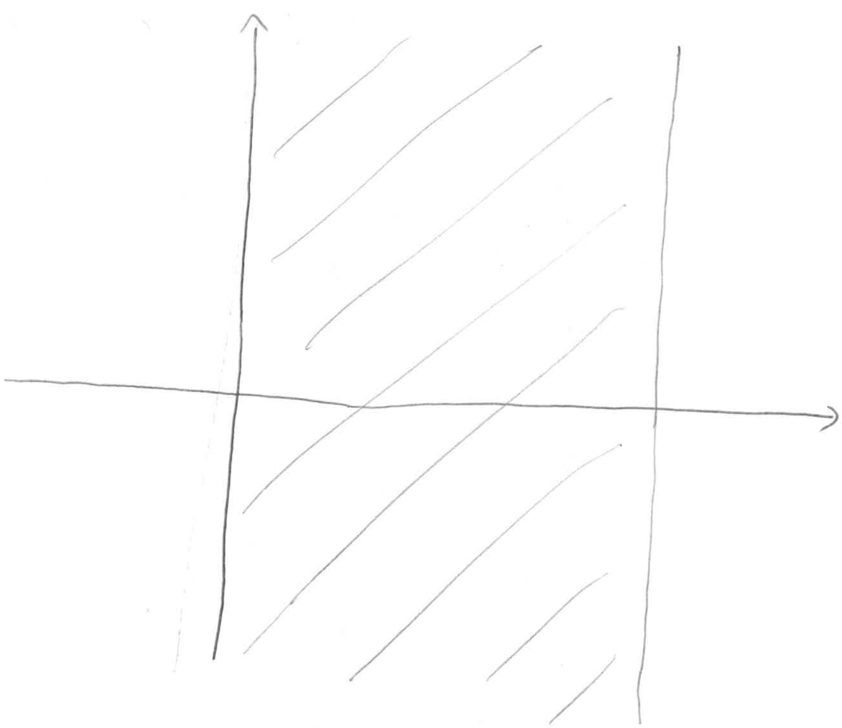
that we get the induced map on $H_q(F)$. A naturality argument shows this.

General Homology Theories

There was nothing special about ordinary cohomology in our construction except for $H_q(X) = 0$ if $q<0$, and $H_{p+q}(X_p, X_{p-1}) = 0$ for $q<0$ for lifts of skeleta.

Given $F \to E \to B$ w/ the usual assumptions, and one's favorite homology theory, one obtains a spectral sequence with $E^2_{p,q} \approx H_p(B; h_q(F))$ which (assuming it converges) converges to $h_*(E)$, and likewise for cohomology.

In particular, $\cdot \hookrightarrow X \to X$ gives a spectral sequence w/ $E^2_{p,q} \approx H_p(X; h_q(X))$



In principle a first & fourth quadrant spectral sequence.

Certainly converges if X is finite dimensional.

(In practice converges much more often than that.)

Rational Homotopy Theory

Let $\mathcal{C}$ be one of

- $\mathcal{FG}$ — Finitely generated abelian groups
- $\tilde{\mathcal{C}}_P$ — P a set of primes, abelian groups w/ order$(g) \in P^n$ for some n for all $g \in G$
- $\mathcal{F}_P$ — Finitely generated (abelian) groups in $\tilde{\mathcal{C}}_P$ (necessarily finite).

(Serre's mod-$\mathcal{C}$ Hurewicz thm)

Theorem 1 Let X be an abelian topological space ($\pi_1(X)$ acts trivially on all $\pi_n(X)$). Then $\pi_k(X) \in \mathcal{C} \iff H_k(X) \in \mathcal{C}$ as well $\forall\ k > 0$.

"Contractible modulo $\mathcal{C}$"

Example 1 $X = S^n$. We know $H_k(S^n) \in FG \Rightarrow \pi_k(S^n) \in FG$ for all k.

Example 2 The assumption that X is abelian is necessary. Consider $S^1 \vee S^2$. $H_k(S^1 \vee S^2)$ is finitely generated, but $\pi_2(S^1 \vee S^2)$ is in fact infinitely generated.

Universal cover

Example 3 Similarly $\mathbb{RP}^{2n}$. $H_k(\mathbb{RP}^{2n}) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}/2 & k \text{ odd} \\ 0 & k>0 \text{ even} \end{cases}$ but $\pi_{2n}(\mathbb{RP}^{2n}) \simeq \pi_{2n}(S^{2n}) \simeq \mathbb{Z} \notin \mathcal{T}_2$

More generally Let $\mathcal{C}$ be any class of group st

1. If $0 \to A \to B \to C \to 0$ is exact, then $B \in \mathcal{C} \iff A, C \in \mathcal{C}$
2. $A, B \in \mathcal{C} \Rightarrow A \otimes B \in \mathcal{C}$, $\mathrm{Tor}_1(A,B) \in \mathcal{C}$

Such a $\mathcal{C}$ is called a Serre class.

Thm 2 X abelian, $\pi_k(X) \in \mathcal{C}$ for $1 \le k < n$. Then $H_k(X) \in \mathcal{C}$ for $1 \le k < n$ and moreover, $h: \pi_n(X) \to H_n(X)$ is an isomorphism modulo $\mathcal{C}$

(i.e., the kernel and cokernel are both in $\mathcal{C}$)

Note that, for example, "isomorphism modulo $\mathcal{T}$" means the map becomes an isomorphism after tensoring with $\mathbb{Q}$.

More Vocabulary

A map $f: X \longrightarrow Y$ (abelian spaces) is a <u>quasi-isomorphism modulo $\mathcal{C}$</u> if $f_*: H_*(X) \to H_*(Y)$ is an isomorphism mod $\mathcal{C}$. X, Y are homotopy equivalent modulo $\mathcal{C}$ if they can be connected by a quasi-isomorphism mod $\mathcal{C}$. (e.g. <u>rational homotopy equivalence</u> is modulo $\mathcal{T}$)

Proof of Thm 1

<u>Lemma</u> $F \to X \to B$ a fibration, all spaces abelian, $\pi_1(B)$ acts trivially on the homology of the fibre. If any two of spaces have homology groups in $\mathcal{C}$ for $k \geq 1$, so does the third.

<u>Pf</u> <u>Case I</u> F, B have homology groups in $\mathcal{C}$. Have spectral sequence converging to $H_*(X)$ w/

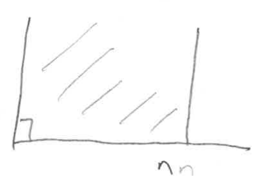
$$E^2_{p,q} = H_p(B; H_q(F)) \simeq \underbrace{(H_p(B) \otimes H_q(F))}_{\substack{\in \mathcal{C} \\ \text{for } (p,q) \neq (0,0)}} \oplus \underbrace{\operatorname{Tor}(H_{p-1}(B), H_q(F))}_{\substack{\in \mathcal{C} \\ \text{for } (p,q) \neq (0,0)}}$$

So if $(p,q) \neq (0,0)$, $E^2_{p,q} \in \mathcal{C}$. But then $E^3_{p,q} = \ker(d^2)/\operatorname{im}(d^2) \in \mathcal{C}$ by the ses condition.

$\rightsquigarrow E^\infty_{p,q} \in \mathcal{C} \rightsquigarrow H_k(X) \in \mathcal{C}$.

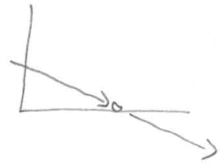
<u>Case II</u> X, F have homology groups in $\mathcal{C}$. We know $E^\infty_{p,q} \in \mathcal{C}$ for $(p,q) \neq (0,0)$

Suppose we know that $H_k(B) \in \mathcal{C}$ for $1 \leq k \leq n$.



At least the shaded part of the quadrant

Notice that the part of the spectral sequence known to be in $\mathcal{C}$ has to stay in $\mathcal{C}$ on subsequent pages. Now, by assumption $E^\infty_{n,0} \in \mathcal{C}$, but $E^\infty_{n,0} = E^{n+1}_{n,0}$, since we must have stabilized at $(n,0)$ by this page



Ergo, we can now use

$d^r : E^r_{n,0} \longrightarrow E^r_{n-r,r-1}$ to produce an ses

$$0 \longrightarrow E^{r+1}_{n,0} \longrightarrow E^r_{n,0} \longrightarrow \underbrace{\mathrm{Im}(d^r)}_{\substack{\cup\!\!| \\ E^r_{n-r,r-1}}} \longrightarrow 0 \qquad \Big\} \in \mathcal{C}$$

to say $E^{r+1}_{n,0} \in \mathcal{C} \Leftrightarrow E^r_{n,0}$ is. So $E^{n+1}_{n,0} \in \mathcal{C} \Rightarrow E^2_{n,0} \in \mathcal{C} \Rightarrow$ $H_n(B; H_0(F)) \in \mathcal{C} \Rightarrow H_n(B) \in \mathcal{C}$. $\square$

<u>Case III</u> $H_*(X)$, $H_*(B)$ are in $\mathcal{C}$. Extremely similar (exercise).

<u>Lemma</u> $G \in \mathcal{C} \Rightarrow H_k(K(G,n)) \in \mathcal{C}$ for $k \geq 1$.

<u>PF</u> Path-loop Fibration

$$\begin{array}{ccc} K(G,n) & \longrightarrow & PK(G,n+1) \\ & & \downarrow \\ & & K(G,n+1) \end{array}$$

$\rightsquigarrow H_*(K(G,n)) \in \mathcal{C}$

$\Leftrightarrow H_*(K(G,n+1)) \in \mathcal{C}$.

Consider $K(G,1)$.

- If $G = \mathbb{Z}$, $H_*(S^1)$ is finitely generated.

- If $G = \mathbb{Z}/n\mathbb{Z}$, $K(\mathbb{Z}/n\mathbb{Z}, 1) = S^\infty/(\mathbb{Z}/n\mathbb{Z})$ an infinite lens space.

$$H_k(K(\mathbb{Z}/n\mathbb{Z},1)) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}/n\mathbb{Z} & k \text{ odd} \\ 0 & k \text{ even} \end{cases} \Big\} \text{ all finite}$$

- Now let $\mathcal{C} = \mathcal{FG}$. $G \in \mathcal{FG} \Rightarrow G = \prod_i (\mathbb{Z}/n_i\mathbb{Z})$. $K(G,1) = \prod_i K(\mathbb{Z}/n_i\mathbb{Z}, 1)$
$\Rightarrow K(G,1)$ has finitely generated homology.

- $\mathcal{C} = \mathcal{F}_p$ is similar

- $\mathcal{C} = \mathcal{T}_p$: Canonical $\overset{BG}{\overset{\shortparallel}{K(G,1)}}$ constructed as the simplicial complex $[g_1|\dots|g_k]$ where $g_i \in G$, as we did in class. If H is a subgroup of G, there is a canonical map $K(H,1) \hookrightarrow K(G,1)$. Now for any $[\alpha] \in H_*(K(G,1))$, α maps to finitely many simplices in $K(G,1) \Rightarrow \alpha \in \operatorname{im}(K(H,1) \hookrightarrow K(G,1))$ for H a finitely generated subgroup. So $[\alpha] = \iota_*[\alpha']$ for some $[\alpha'] \in H_*(K(H,1))$. $[\alpha']$ has torsion of order divisible by $p \Rightarrow$ the same is true of $[\alpha]$.

<u>Next Time</u> Proof of Thm 1