Lecture 15

Products on the cohomological Serre spectral sequence.

Goal: Compute $H^*(U(n))$, $H^*(BU(n))$, at least over a field.

Thm: In the cohomology Serre spectral sequence for $F \to E \to B$ with B simply connected,

1. $\forall$ n the E_n page has a product $E_n^{p,q} \times E_n^{s,t} \longrightarrow E_n^{p+s,q+t}$

   For $n=2$: $H^p(B; H^q(F)) \times H^s(B; H^t(F)) \longrightarrow H^{p+s}(B; H^{q+t}(F))$

   via $(-1)^{qs}$ times the cup product on cycles & coefficients

   or over a field, $H^p(B) \otimes H^q(F) \times H^s(B) \otimes H^t(F) \longrightarrow H^{p+s}(B) \otimes H^{q+t}(F)$

   $(\alpha, \beta) \qquad (x, y) \qquad (-1)^{qs}(\alpha \cup x) \otimes (\beta \cup y)$

   (This is the usual graded tensor product that typically appears in the Kunneth formula.)

2. The maps d_n respect the product in the sense that

   $d_n(\alpha \cdot \beta) = d_n(\alpha) \cdot \beta + (-1)^{|\alpha|} \alpha \cdot d_n(\beta)$.

$\rightsquigarrow$ 3. Product on E_n induces the product on E_{n+1}

The product on the E_∞-page may fail to be the product on H^*; in general, we get the induced product on $gr(H^*)$, which even over a field may not determine the original.

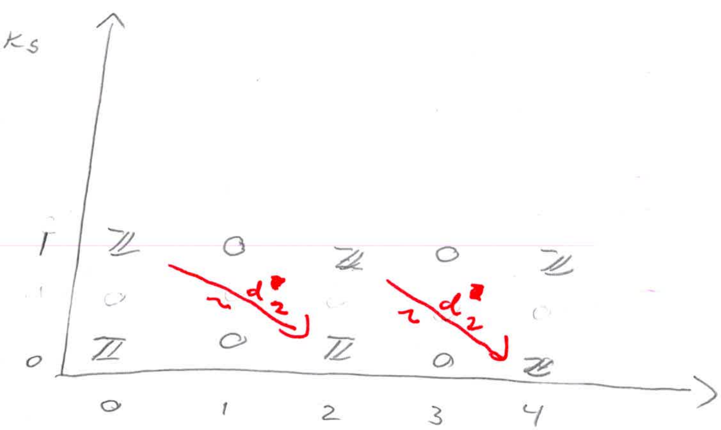
Note: If we don't insist on B simply-connected there is an extra annoying technical issue.

Proof next time: Today, examples and computations.

Example 1 Recall we used homology spectral sequence to (re)compute $H_*(\mathbb{CP}^\infty)$. Now we can (re)determine $H^*(\mathbb{CP}^\infty; \mathbb{Z})$.

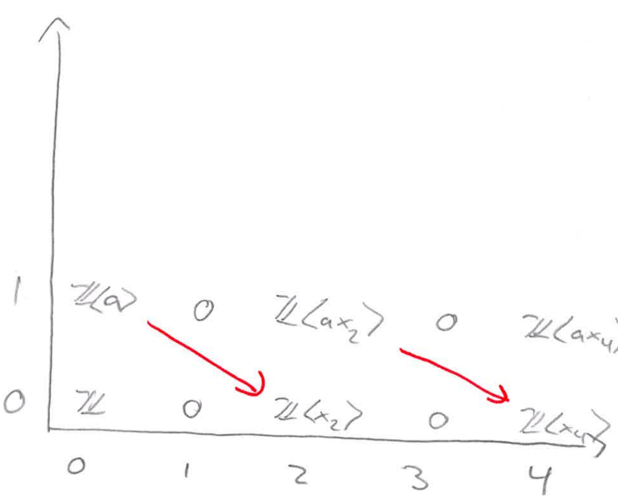
$$S^1 = K(\mathbb{Z},1) \longrightarrow PK(\mathbb{Z},2)$$
$$\downarrow$$
$$K(\mathbb{Z},2) = \mathbb{CP}^\infty$$

E_2 page looks like



Now let's add structure. $H^*(S^1; \mathbb{Z}) = \mathbb{Z}[a]/(a^2)$. Let x_2, x_4, x_6, etc be generators along the bottom row so that x_i is in degree i (so as a module w/o a multiplication, $H^*(\mathbb{CP}^\infty; \mathbb{Z})$ must be $\mathbb{Z} \oplus \mathbb{Z}x_2 \oplus \mathbb{Z}x_4 \oplus \cdots$)

Now, the copy of $\mathbb{Z}$ at $(i,1)$ must be generated by ax_i because the product $E_2^{0,q} \times E_2^{s,t} \longrightarrow E_2^{s,q+t}$ is multiplication of coefficients.



Up to switching signs, $d_2(a) = x_2$.

So $d_2(ax_{2i}) = (d_2 a)x_{2i} \pm a\underset{0}{d_2(x_{2i})}$

$= (d_2 a)x_{2i}$

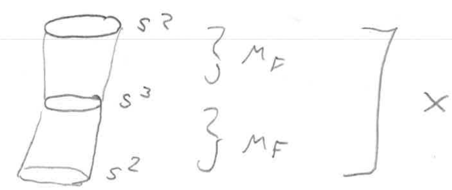
$= \underbrace{x_2 x_{2i}}$

This has to generate b/c d_2 is an isomorphism. So → etc we see that $x_2 x_{2i} = x_{2i+2}$ up to isomorphism. So along the bottom row we have $H^*(\mathbb{C}P^\infty; \mathbb{Z}) = \mathbb{Z}[x_2]$.

<u>Example 2</u> Product on E_∞ page versus product on H^*.

$$\begin{array}{ccc} S^1 & \hookrightarrow & S^3 \\ & & \downarrow F \\ & & S^2 \end{array}$$ be the Hopf bundle. Take two copies of M_F

and glue them together along S^3.



Each mapping cylinder is a bundle $$\begin{array}{ccc} D^2 & \hookrightarrow & M_F \\ & & \downarrow \\ & & S^2 \end{array}$$ [It's $S^3 \times I$ with F done to one end, sealing each copy of $S^1 \times I$ in the center.]

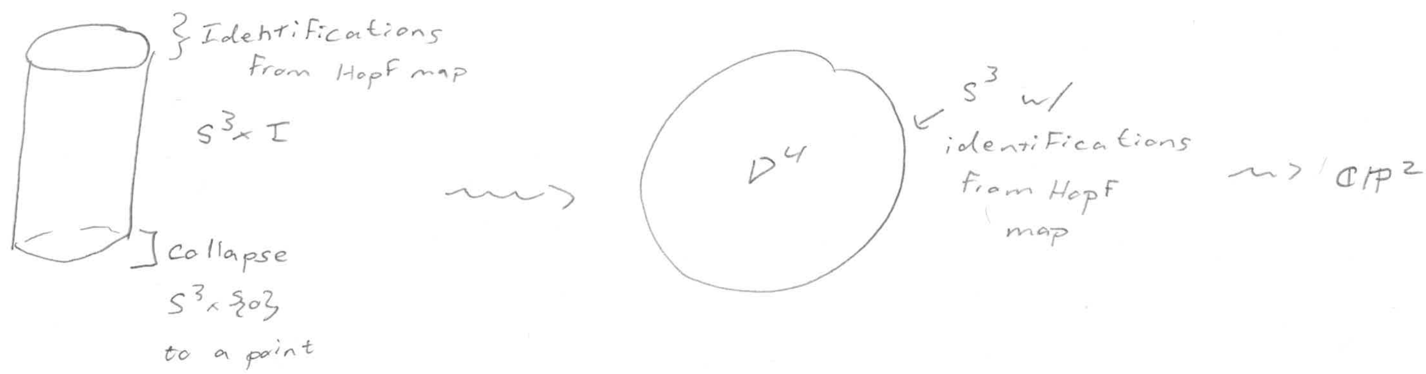
So in total we have a bundle $$\begin{array}{ccc} S^2 & \hookrightarrow & X \\ & & \downarrow \\ & & S^2 \end{array}$$

$E_2 = E_\infty$ page of the Serre spectral sequence is:

There are no nontrivial differentials ∴ the product structure never changes.

$$\begin{array}{c|ccc} 2 & \mathbb{Z}\langle a\rangle & 0 & \mathbb{Z}\langle ab\rangle \\ 1 & 0 & 0 & 0 \\ 0 & \mathbb{Z} & 0 & \mathbb{Z}\langle b\rangle \\ \hline & 0 & 1 & 2 \end{array}$$

<u>However</u> If we collapse one copy of M_f we have



So we have a map $q: X \to \mathbb{C}P^2$. $q^*: H^4(\mathbb{C}P^4) \to H^4(X)$ is an isomorphism by considering the les for the pair (X, M_f) and remembering that $M_f \simeq S^2$. So $q^*: H^2(\mathbb{C}P^\infty) \to H^2(X)$ takes the generator of $H^2(\mathbb{C}P^\infty)$ to something squaring to a generator of $H^4(\mathbb{C}P^\infty)$. But $S^2 \times S^2$ has no such classes in its cohomology.

<u>Thm</u> $H^*(SU(n); k) \simeq \Lambda(x_3, x_5, \dots, x_{2n-1})$ with $\deg(x_i) = i$.

↑ Free graded commutative algebra

<u>PF</u> <u>Base Cases</u> $n=1$ $SU(1) = \{1\}$ $H^*(SU(1)) = k$

$n=2$ $SU(2) = S^3$ $H^*(SU(2)) = \Lambda(x_3)$

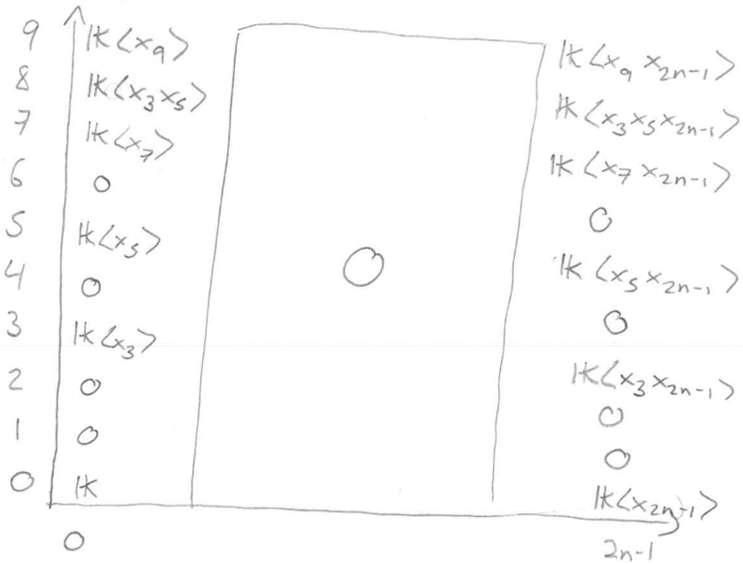
<u>Inductive Step</u> There is a fiber bundle

$$\begin{array}{ccc} SU(n-1) & \longrightarrow & SU(n) \\ & & \downarrow \\ & \text{simply-ctd} \checkmark & S^{2n-1} \end{array}$$

$\rightsquigarrow$ spectral sequence w/ $E_2^{p,q} = H^p(S^{2n-1}; k) \otimes H^q(SU(n-1); k)$.

As a ring, $E_2^{p,q} = \Lambda(x_{2n-1}) \underset{\text{graded}}{\otimes} \Lambda(x_3, \ldots, x_{2n-3}) = \Lambda(x_3, \ldots, x_{2n-3}, x_{2n-1})$

The E_2 page looks like



Only possible differential occurs on page E_{2n-1}. But we see that

$d_{2n-1}(x_3) = d_{2n-1}(x_5) = \cdots = d_{2n-1}(x_{2n-3}) = 0$

(they land in the second or fourth quadrants).

Similarly we see $d_{2n-1}(x_{2n-1}) = 0$

Ergo we conclude that $d_{2n-1} \equiv 0$. So we know the product structure on the E_∞ page.

Claim This is also the product structure on H^*.

Lemma Let A be a filtered graded algebra

$$A = A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$$

st $A_iA_j \subseteq A_{i+j}$, so that $gr(A)$ is a free graded algebra. Then A itself is a free graded algebra.

PF Let B be a free graded algebra w/ gradings $x_1, \ldots, x_n$ in the same gradings as $a_1, \ldots, a_n$ generators of $gr(A)$, and a filtration such that $A_n/A_{n-1} \cong B_n/B_{n-1}$. There is a filtered ring homomorphism $B \to A$. Suppose there is a relation among the

a_i, i.e. $\exists$ R st $R \mapsto 0$ under $B \to A$. Look at the largest n such that $R \in B_n$. Then $B_n \to A_n \to A_n/A_{n-1} \simeq B_n/B_{n-1}$. But

$$R \mapsto 0 \mapsto 0 \mapsto 0 = [R]$$

$[R] \neq 0$ in B_n/B_{n-1}. So $\#$.

$\rightsquigarrow$ Ergo $H^*(SU(n);\mathbb{K}) \simeq \Lambda(x_3, \dots, x_{2n-1})$.

<u>Note</u> The reason this works here and not in Example 2 is gradings; $H^*(S^2 \times S^2)$ has $a^2 = 0$ even though a is in <u>even</u> degree.

<u>Corollary</u> $H^*(U(n);\mathbb{K}) \simeq \Lambda(x_1, \dots, x_{2n-1})$.

<u>Proof</u> Topologically, $U(n) \simeq S^1 \times SU(n)$. Cohomology of products $\rightsquigarrow$ graded tensor product of rings.

Chern classes. We'll be seeing these...

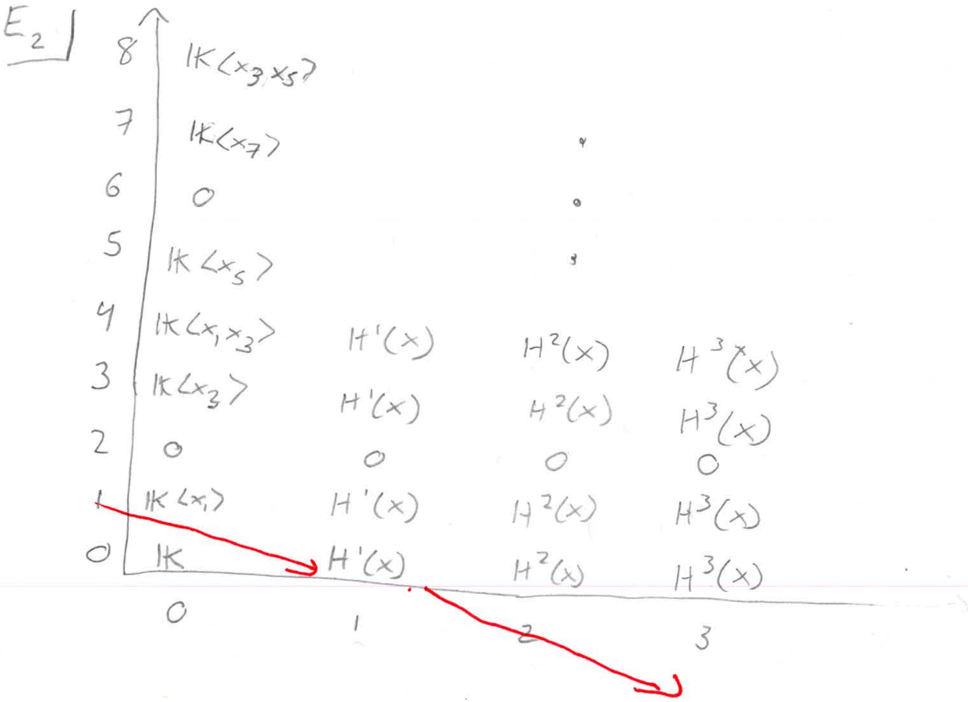
<u>Thm</u> $H^*(BU(n);\mathbb{K}) \simeq \mathbb{K}[c_1, c_2, \dots, c_n]$ where $\deg(c_i) = 2n$.

<u>Proof</u>

<u>Case I</u> $n = 1$, $BU(1) = Gr_1(\mathbb{C}^\infty) = \mathbb{CP}^\infty$. [$U(1)$ acts on $EU(1) = S^\infty$, quotient is $BU(1) = \mathbb{CP}^\infty$.]

<u>Inductive Step</u> Consider

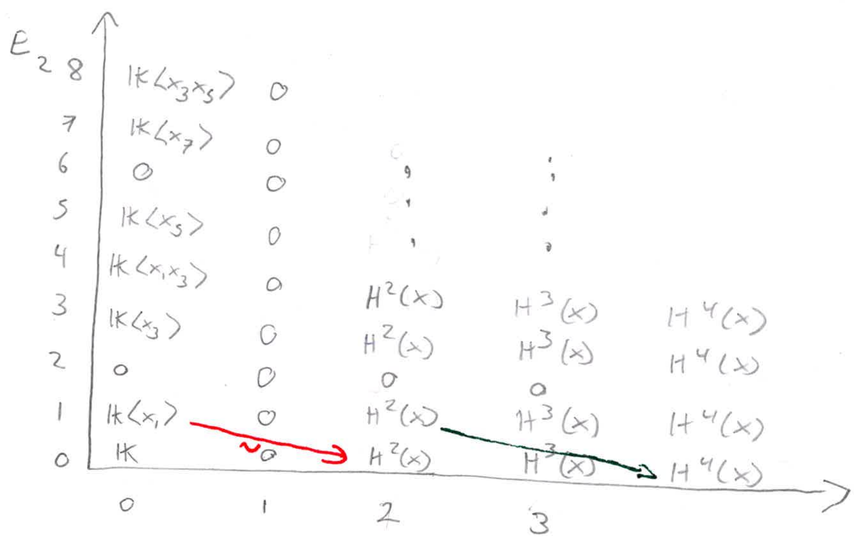
$$\begin{array}{ccl} U(n) & \longrightarrow & EU(n) \; \} \text{ contractible} \\ & & \downarrow \\ & & BU(n) \; \} \text{ simply ctd} \end{array}$$



Let $X = BU(n)$.

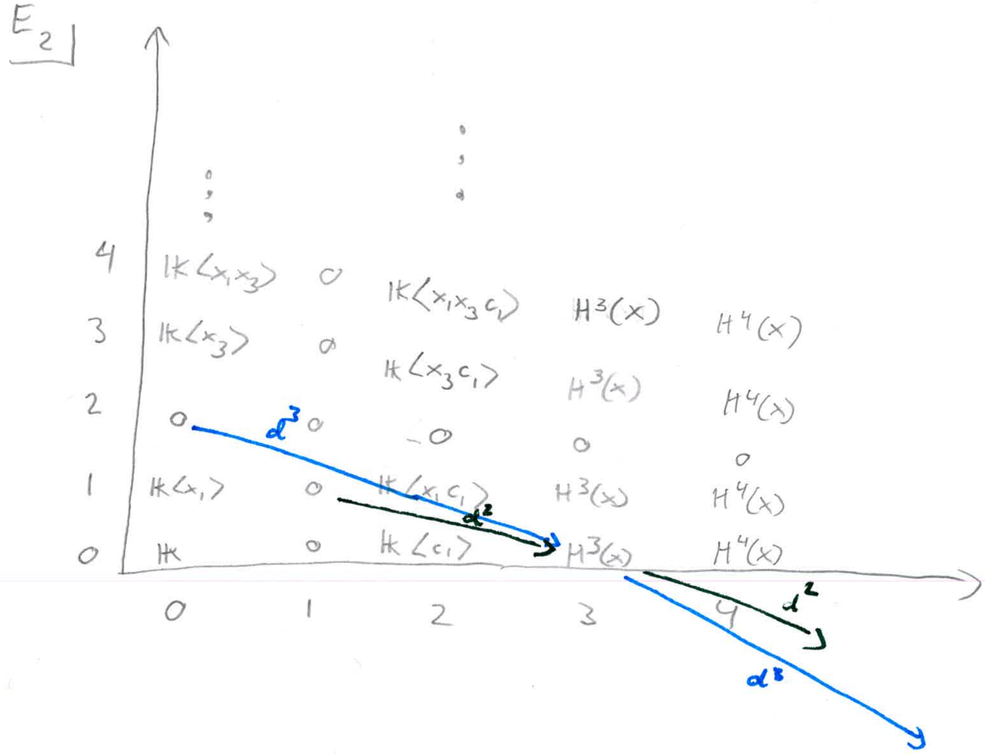
Note there is no way to kill the copy of $H^1(X)$ at $(1,0)$. So $H^1(X) = 0$.

$\rightsquigarrow$



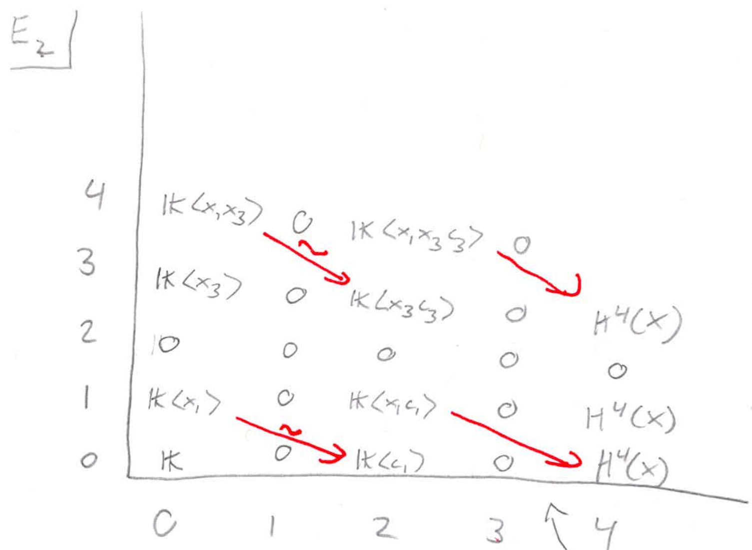
The copy of $H^2(X)$ at $(2,0)$ can't die except by having the red map shown be an isomorphism. So $d_2(x_1) = c_1$, where $H^2(X) = \mathbb{K}\langle c_1\rangle$

$\Rightarrow d_2(x_1 c_1) = (d_2 x_1)c_1 \pm \cancel{x_1 d_2 c_1}^{\,0} = c_1^2$. Since the green map has to be an isomorphism in order for the copy of $H^2(X)$ at $(2,1)$ to die, we conclude $c_1^2 \neq 0$ in $H_4(X)$.



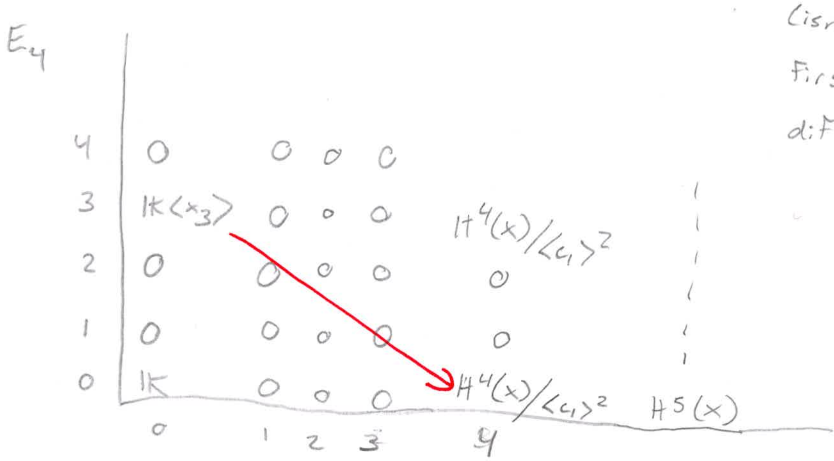
We see the copy of $H^3(X)$ at $(3,0)$ can't be killed by a d^2 or d^3 differential, so $H^3(X)=0$

So we have



There are no d_3 differentials in this part of the spectral sequence, and we can go straight to E_4.

Note that this one may not be (isn't, in fact) an isomorphism. Our first example of a nonzero, nonisomorphism differential!



Can now argue $H^5(X)=0$.

Furthermore,

$d_4 : \Bbbk\langle x_3 \rangle \xrightarrow{\sim} H^4(X)/\langle c_1^2 \rangle$ so $d_4(x_3) \neq 0$ in $H^4(X)/\langle c_1^2 \rangle$. Call

$d_4(x_3) = c_2$. So $H^4(X) = \Bbbk\langle c_1, c_2 \rangle$.

Now tracing back to the E_2 page, we see that $d_2 : H^4(X) \otimes \langle x_1 \rangle \xrightarrow{\sim} H^6(X)$

$\Rightarrow c_1^3 \neq 0$, $c_1 c_2 \neq 0$ $\Rightarrow$ all products $c_1^k c_2^j$ are nonzero ~~after~~ further similar arguments.

A version of the previous lemma for free (ungraded) algebras shows this is really the ring structure on $H^*(BU(n); \Bbbk)$.