Lecture 14

Recall from last time Exact couple

$$\begin{tikzcd} A^0_{*,*} \arrow[rr, "{[1,-1]}", "i"'] & & A^0_{*,*} \arrow[dl, "j"', "{[0,0]}"] \\ & E^0_{*,*} \arrow[ul, "k", "{[0,-1]}"'] & \end{tikzcd}$$

$\rightsquigarrow$ Derived Couple

$$\begin{tikzcd} A^1_{*,*} \arrow[rr, "{[1,-1]}", "i'"'] & & A^1_{*,*} \arrow[dl, "j'"', "{[-1,1]}"] \\ & E^1_{*,*} \arrow[ul, "k'", "{[0,-1]}"'] & \end{tikzcd}$$

- $A^1_{*,*} = i(A^0_{*,*})$
- In particular, $A^1_{p,q} = i(A^0_{p-1,q+1})$
- $E^1_{*,*} = H_*(E_{*,*}, j\circ k)$

Convenient simplifying assumptions

- $A^1_{p,q} = 0$ for $p \ll 0$
- $E_{p,q}$ is first quadrant

$\Rightarrow$ $i : A^r_{p,q} \longrightarrow A^r_{p+1,q-1}$ becomes an isomorphism for $p \gg 0$.

At the rth step, we have an les

$$\longrightarrow E^r_{p+r-1,q-r+2} \longrightarrow A^r_{p+r-1,q-r+1} \longrightarrow A^r_{p+r,q-r} \longrightarrow E^r_{p,q} \longrightarrow A^r \longrightarrow$$

- However $E^r_{p+r-1,\, q-r+2} = 0$ for $r > q+2$.

- Furthermore, $A^r_{p,q-1} = i^{r-1}(A^1_{p-r,\, q-r}) = 0$ for $r >> 0$.

So eventually we have an ses

$$0 \longrightarrow A^r_{p+r-1,\, q-r+1} \xrightarrow{i} A^r_{p+r,\, q-r} \longrightarrow E^r_{p,q} \longrightarrow 0$$

All these things have to stabilize eventually.

Let $F_{p,q} = \lim_{r \to \infty} A^r_{p+r,\, q-r}$. $F_{p,q}$ is the image of $A^1_{p,q}$ in $A^\infty = i^\infty(A)$.

Then $E^\infty_{p,q} = F_{p,q} / F_{p-1,\, q+1}$.

In the case we care about: we start with

$\phi \subseteq X_1 \subseteq X_2 \subseteq \dots \subseteq X$ a filtration of a topological space.

We have a filtration $\{0\} \subseteq C_*(X_1) \subseteq C_*(X_2) \subseteq C_*(X_3) \subseteq \dots \subseteq C_*(X)$

We look at

$$\begin{tikzcd}
\bigoplus_{p,q} C_{p+q}(X_{p-1}) \arrow[rr, hook, "i"] & & \bigoplus C_{p+q}(X_p) \arrow[dl] \\
& \bigoplus_{p+q} C_{p+q}(X_p) / C_{p+q}(X_{p-1}) \arrow[ul] &
\end{tikzcd}$$

We replace the ses w/ an les on homology

$$\begin{tikzcd}[column sep=small]
\bigoplus_{p,q} H_{p+q}(X_{p-1}) \arrow[rr, "i_*"] & & \bigoplus_{p,q} H_{p+q}(X_p) \arrow[dl] \\
& \bigoplus_{p,q} H_{p+q}(X_p, X_{p-1}) \arrow[ul, "\text{Connecting homomorphism}"] &
\end{tikzcd}$$

Notice that $A^1_{p-1,q+1} \xrightarrow{i_*} A^1_{p,q}$

$$H_{p+q}(X_{p-1}) \longrightarrow H_{p+q}(X_p)$$

eventually stabilizes if $X = \bigcup_p X_p$ w/ the weak topology

So $F_{p,q}$ is the image of $H_{p+q}(X_p) \hookrightarrow H_{p+q}(X)$. [Anything interesting immediate propagates upward, and we're left with just that image.]

We see $E^\infty_{p,q} = F_p H_{p+q}(X) \big/ F_{p-1} H_{p+q}(X)$ so the E^∞ page is the associated graded of $H^*(X)$.

<u>Exercise</u> More generally, we see

$$E^\infty_{p,q} = \frac{k^{-1}(\cap \operatorname{Im}(i^r))}{j(\bigcup_r \ker(i^r))}.$$

If $\bigcap_r \operatorname{Im}(i^r) = 0$ we can recover the ses above.

Thm $F \to X \xrightarrow{\pi} B$ a fibration, B acts trivially on $H_*(F;R)$

There is a spectral sequence converging to $H_*(X;R)$ with

$$E^2_{p,q} = H_p(B; H_q(F;R)).$$

Pf Filtration by $X_p = \pi^{-1}(B^p)$.

Exercise If (X, Y) is n-connected, and $E \to X$ is a fibration, then $(E, \pi^{-1}(Y))$ is n-connected.

We will show ① $E^1_{p,q} \overset{\text{def}}{=} H_{p+q}(X_p, X_{p-1}; R) = H_p(B^p, B^{p-1}; \mathbb{Z}) \otimes H_q(F;R)$

One also needs to check

② d^1 is $\partial \otimes Id : H_p(B^p, B^{p-1}; \mathbb{Z}) \otimes H_q(F;R)$

$$\downarrow$$

Cellular differential

$H_{p-1}(B^{p-1}, B^{p-2}; \mathbb{Z}) \otimes H_q(F;R)$

Proof of ① B^P has p-cells e^p_α, $\alpha \in A$. For each α, consider

$$\begin{tikzcd}
\tilde{S}^{p-1}_\alpha \arrow[r, hook] \arrow[d] & \tilde{D}^p_\alpha \arrow[r] \arrow[d] & X^P \arrow[d] \\
\partial e^p_\alpha \arrow[r, hook] & e^p_\alpha \arrow[r] & B^P
\end{tikzcd}$$

Pullback Fibration

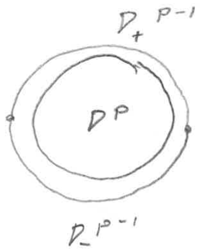
Claim $H_{p+q}(X_p, X_{p-1}) = \bigoplus_\alpha H_{p+q}(\tilde{D}^p_\alpha, \tilde{S}^{p-1}_\alpha)$.

Pf Replace B^{p-1} by a regular neighborhood N. Use excision to remove $\pi^{-1}(N)^\circ$.

<u>Lemma</u> D^p a disk, $S^{p-1} = \partial D^p$, $F \to \tilde{D}^p \to D^p$ a fibration. Then

$H_{p+q}(\tilde{D}^p, \tilde{S}^{p-1}; R) \simeq H_q(F; R)$.

<u>Pf</u>



$(D^p, S^{p-1}, D_-^{p-1})$

$$H_{p+q}(\tilde{D}^p, \tilde{S}^{p-1}) \xrightarrow{\simeq} H_{p+q-1}(\tilde{S}^{p-1}, \tilde{D}_-^{p-1})$$

because $H_*(\tilde{D}^p, \tilde{D}^{p-1}) = 0$ by lifting the homotopy $D^p \to D_-^{p-1}$ to $\tilde{D}^p \to \tilde{D}_-^{p-1}$.

So

$$\begin{tikzcd} H_{p+q}(\tilde{D}^p, \tilde{S}^{p-1}) \arrow[rr, "\sim"] \arrow[dr, "\simeq"'] & & H_{p+q-1}(\tilde{D}_+^{p-1}, \tilde{S}^{p-2}) \\ & H_{p+q-1}(\tilde{S}^{p-1}, \tilde{D}_-^{p-1}) \arrow[ur, "\text{excision}"'] & \end{tikzcd}$$

Iterating, we have $H_{p+q}(\tilde{D}^p, \tilde{S}^{p-1}) \simeq H_{q+1}(\tilde{D}_+^1, \tilde{S}^0) \simeq H_q(\tilde{D}_+^0) \simeq H_q(F; R)$

We now know the vector spaces on the E^1 page of the spectral sequence.

<u>This implies that</u> $H_{p+q}(X_p, X_{p-1}) = \bigoplus_\alpha H_{p+q}(\tilde{D}_\alpha^p, \tilde{S}_\alpha^{p-1})$

$= \bigoplus_\alpha H_q(F; R)$

$= H_p(B^p, B^{p-1}) \otimes H_q(F; R)$

Note that we are implicitly using here that all copies of $H_q(F; R)$

are canonically identified; i.e, that the action of $\pi_1(B)$ on $H_*(F;R)$ is trivial.

<u>Proposition</u> $d^1: H_{p+q}(X_p, X_{p-1}) \longrightarrow H_{p+q-1}(X_{p-1}, X_{p-2})$

$$\begin{tikzcd} H_{p+q}(X_p, X_{p-1}) \arrow[rr, "d^1"] \arrow[dr, "j'"'] & & H_{p+q-1}(X_{p-1}, X_{p-2}) \\ & H_{p+q-1}(X_{p-1}) \arrow[ur, "k'"'] & \end{tikzcd}$$

agrees with $\partial \otimes \mathrm{Id}$, where ∂ is the cellular cohomology differential. That is to say, it is given by the degrees of the attaching maps of the p-cells in B.

<u>PF</u> A diagram chasing exercise extraordinarily similar to what we did for cohomology theories.

<u>What do we do for cohomology?</u>

We need to add the assumption that $H^n(X, X_p) = 0$ for large p.

We set $A_{-p,-q} = H^{p+q}(X, X_{p-1})$

$$E_{-p,-q} = H^{p+q}(X_p, X_{p-1})$$

Long exact sequence for the triple gives an exact couple

$$\begin{tikzcd} A_{*,*} \arrow[rr] & & A_{*,*} \arrow[dl] \\ & E_{*,*} \arrow[ul] & \end{tikzcd}$$

with the same bidegrees as previously (because of the minus signs).

$H^*(X)$ has a Filtration by

$$F_{-p} = \mathrm{Im}\left(H^{p+q}(X, X_p) \longrightarrow H^{p+q}(X)\right) = \ker\left(H^{p+q}(X) \to H^{p+q}(X_p)\right)$$

Flipping the signs of p and q gives an descending Filtration $H^*(X) \supseteq F^0 \supseteq F^1 \supseteq F^2 \supseteq \ldots$

and a cohomological spectral sequence with $E_1^{p,q} = H^{p+q}(X_p, X_{p-1})$ & cellular cohomology differentials.

<u>Next Goal</u> Compute $H^*(U(n))$

$H^*(BU(n))$

<u>Multaplicative properties of cohomology Serre sequences</u>

① $\forall n$ the E_n page has a product

$$E_n^{p,q} \times E_n^{r,s} \longrightarrow E_n^{p+r, q+s}$$

For $n=2$, over a Field $E_2^{p,q} = H^p(B) \otimes H^q(F)$, and the product is $(a \otimes b)(x \otimes y) = (a \cup x) \otimes (b \cup y)$

② The maps d_n respect the product in the sense that

$$d_n(\alpha \cdot \beta) = (d_n(\alpha)) \cdot \beta + (-1)^{|\alpha|} \alpha \cdot d_n(\beta)$$

$\Rightarrow$ ③ Product on E_n induces the product on E_{n+1}

Reminder The product on the E_∞ page may not be the product on H^*.

Thm $H^*(SU(n);\mathbb{k}) \simeq \Lambda(x_3, x_5, \dots, x_{2n-1})$ with $\deg(x_i) = i$.

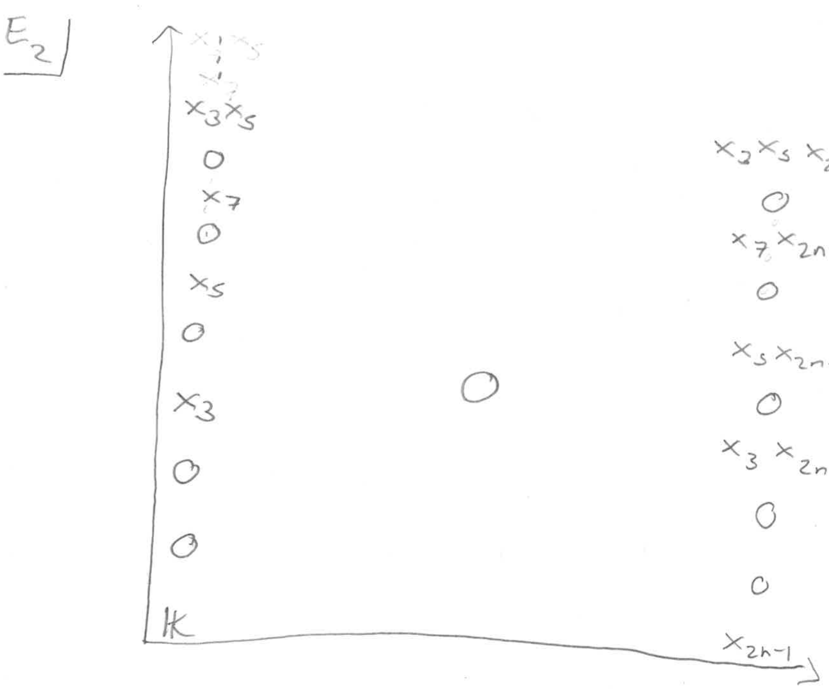
PF Base Case(s) $n=1$ $SU(1) = \{1\}$ $H^* = \mathbb{k}$

$n=2$ $SU(2) = S^3$, $H^* = \Lambda(x_3)$

Induction Step We use the fibre bundle

$$\begin{array}{ccc} SU(n-1) & \longrightarrow & SU(n) \\ & & \downarrow \\ & & S^{2n-1} \end{array}$$

$\rightsquigarrow$ spec. seq. with $E_2^{p,q} = H^p(S^{2n-1}) \otimes H^q(SU(n-1))$. As a ring, $E_2^{p,q} = \Lambda(x_{2n-1}) \otimes \Lambda(x_3, \dots, x_{2n-3}) = \Lambda(x_3, \dots, x_{2n-1})$. All generators live in (p,q) degrees w/ p,q odd. We have



Only possible differential is d_{2n-1}. But

$d_{2n-1}(x_3) = d_{2n-1}(x_5)$

$= \dots = d_{2n-1}(x_{2n-3}) = 0$

and $d_{2n-1}(x_{2n-1}) = 0$,

so $d_{2n-1} \equiv 0$.

Next Time Ring Structure.