Lecture 13

Recall from last time A first quadrant spectral sequence of homological type over $\mathbb{k}$ consists of $\mathbb{k}$-vector spaces $E^r_{p,q}$ and maps $d^r_{p,q} : E^r_{p,q} \longrightarrow E^r_{p-r,\, q+r-1}$ st

- $d^r_{p-r,\, q+r-1} \circ d^r_{p,q} = 0$
- $E^{r+1}_{p,q} = \ker(d^r_{p,q}) / \operatorname{im}(d^r_{p+r,\, q-r+1})$
- $E^r_{p,q} = 0$ if $p < 0$ or $q < 0$.

& likewise for cohomology. We say the spectral sequence converges to $E^\infty_n := \bigoplus_{p+q}^{\infty} E^\infty_{p,q}$ if $E^\infty_{p,q} = E^N_{p,q} = E^{N+1}_{p,q} = \cdots$ for $N >> 0$.

Thm Let $\begin{array}{c} F \hookrightarrow E \\ \downarrow \\ B \end{array}$ a fibration such that $\pi_1(B)$ acts trivially on $H_*(F;\mathbb{k})$. There is a spectral sequence such that $E^2_{p,q}$ is $H_p(F; H_q(B))$ and $\underbrace{E^\infty_n = H_n(E)}$

depends on field, to be upgraded shortly.

Generalizes the Kunneth Formula

If X, Y are CW complexes and R is a principal ideal domain, there are natural short exact sequences

$$0 \longrightarrow \bigoplus_i \left(H_i(X;R) \otimes_R H_{n-i}(Y;R)\right) \longrightarrow H_n(X \times Y; R)$$

$$\longrightarrow \bigoplus_i \mathrm{Tor}_R\left(H_i(X;R), H_{n-i-1}(Y,R)\right) \longrightarrow 0$$

[If necessary: Recall that we compute $\mathrm{Tor}_R(C_*, D_*)$ via taking a projective resolution $\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$, tensoring w/ B and removing the last term $\cdots \longrightarrow P_2 \otimes B \longrightarrow P_1 \otimes B \longrightarrow P_0 \otimes B \longrightarrow 0$ and taking homology. In general $\mathrm{Tor}_R(C_*, D_*)$ measures the difference between $H_*(C_* \otimes D_*)$ and $H_*(C_*) \otimes H_*(D_*)$, and vanishes over a field.

Combining Kunneth & universal coefficients gives (non-canonically)

$$H_n(X \times Y; R) \simeq \bigoplus_i H_i(X; H_{n-i}(Y;R))$$

Likewise for cohomology.

Fundamental Algebra Structures

0. Double Complexes ← Easiest to compute with
1. Filtered complexes
2. Exact Couples ← Most general

Defn A Filtered module over a ring R is a module M w/ either

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq \dots \quad \} \text{ a descending Filtration}$$

or

$$\{0\} = M_0 \subseteq M_1 \subseteq M_2 \subseteq M_3 \subseteq \dots \quad \} \text{ an ascending Filtration}$$

We say $M_i = F_i M$. The example to keep in mind is that if $\{0\} \subseteq X_1 \subseteq X_2 \subseteq \dots \subseteq X$ is inclusions of topological spaces, $\{\phi\} \subseteq C_*(X_1) \subseteq C_*(X_2) \subseteq \dots \subseteq C_*(X)$ is an ascending Filtration on homology (For cohomology we can construct a descending Filtration given more hypotheses.)

Defn Given a Filtered module M, the associated graded module $gr(M)$ has $gr(M)_i = F_i M / F_{i-1} M$ if the Filtration is ascending and

$gr(M)_i = F_i M / F_{i-1} M$ if descending.

[Recall A graded module M is one that decomposes as $\bigoplus_{i \in \mathbb{Z}} M_i$ w/ M_i closed under addition & scalar multiplication, unless R is graded, in which case one wants $R_j M_i \subseteq M_{i+j}$.]

- A Filtration is said to be bounded if it has finitely many steps (so for ascending, $M_n = M$ for some n).

- If $M = V$ is a filtered vector space, $\bigoplus_i gr_i(V) \cong V$.

- Over rings this is an extension problem.

Examples

(1) $\{0\} \subseteq 4\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq \mathbb{Z} = M$

$\mathbb{Z} \quad \mathbb{Z}/2\mathbb{Z} \quad \mathbb{Z}/2\mathbb{Z}$

$\{0\} \subseteq \mathbb{Z} \subseteq (\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z} \subseteq (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \oplus \mathbb{Z}$

$\mathbb{Z} \quad \mathbb{Z}/2\mathbb{Z} \quad \mathbb{Z}/2\mathbb{Z}$

$gr_i(M)$ does not determine M

(2) Algebras:

$$M = \mathbb{Q}[a,b,c] \Big/ \begin{matrix} a^2 = b^2 = c^2 = ac = bc = 0 \\ ab = c \end{matrix}$$

$$M = \mathbb{Q}[a,b,c]/\sim \;\supseteq\; \mathbb{Q}[b,c]/\sim \;\supseteq\; \mathbb{Q}[c]/\sim \;\supseteq\; \{0\}$$

In M, $ab = c$. In the associated graded

$\mathbb{Q}[a]/(a^2) \oplus \mathbb{Q}[b]/(b^2) \oplus \mathbb{Q}[c]/(c^2)$, $ab = 0$.

Thm! Given a bounded filtered chain complex C_* there is a spectral sequence with

$$\left\{ \begin{aligned} E^0_{p,q} &= F^p C_{p+q} / F^{p-1} C_{p+q} \\ E^1_{p,q} &= H_{p+q}\left(F^p C / F^{p-1} C\right) \end{aligned} \right\}$$

which converges to $H_*(C_*)$. This means $E^\infty_{p,q} = F^p H_{p+q}(C_*) \big/ F^{p-1} H_{p+q}(C_*)$.

Spoilers In the case of the Serre spectral sequence, the filtration is via skeleta of B. We have $C_*(E)$ filtered by

$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[d] \\ & B \end{tikzcd}$$

$F_p C_*(E) = C_*(\pi^{-1}(B^p))$. On the E^1 page, this returns $H_{p+q}\left(C_*(\pi^{-1}(B^p)) / C_*(\pi^{-1}(B^{p-1})) \right) \simeq H_p\left(B^p / B^{p-1}; H_q(F) \right)$

This is a cellular cohomology complex for $H_p(B; H_q(F))$.

PF A very messy algebra exercise we won't do, because its implied by the proof of the other way spectral sequences can arise.

Exact Couples DeFn An exact couple is a diagram

$$\begin{tikzcd} A_{*,*} \arrow[rr, "{[1,-1]}", "i"'] & & A_{*,*} \arrow[dl, "j\ {[0,0]}"] \\ & E_{*,*} \arrow[ul, "{[0,-1]}\ K"] & \end{tikzcd}$$

Typically these are bigraded modules & grading shifts are as shown.

eg) $i : A_{p,q} \longrightarrow A_{p+1,q-1}$ $\qquad j \circ k : E_{p,q} \longrightarrow E_{p,q-1}$

$j : A_{p,q} \longrightarrow E_{p,q}$

$k : E_{p,q} \longrightarrow A_{p,q-1}$

Defn The derived couple of an exact couple is $A' = i(A)$, $E' = H(E)$ wrt $j \circ k$,

and

$$\begin{tikzcd} A'_{*,*} \arrow[rr, "{[1,-1]}", "i'"'] & & A'_{*,*} \arrow[dl, "j'"', "{[-1,1]}"] \\ & H(E_{*,*}) \arrow[ul, "k'", "{[0,-1]}"'] & \end{tikzcd}$$

- $i' = i$
- $j'(ia) = [ja]$
- $k'([b]) = kb$

Check

- j' is well-defined:
  - $j \circ k(ja) = j \circ (k \circ j)a = j(0) = 0$, so $[ja] \in H(E)$
  - If $i(a) = i(a')$, then $a - a' \in \ker(i) \Rightarrow a - a' \in \operatorname{im}(k)$ $\Rightarrow \exists\, b \in E$ st $k(b) = a - a'$. Then $j \circ k(b) = ja - ja'$, so $[ja] = [ja']$

- k' is well-defined
  - $j \circ k(b) = 0 \Rightarrow j(kb) = 0 \Rightarrow kb \in \operatorname{im}(i) = i(A) = A'$.
  - $[b] = [b'] \Rightarrow b - b' = j \circ k(c) \Rightarrow kb - kb' = \underbrace{k \circ j}_{0} \circ k(c) = 0$ $\Rightarrow k(b) = k(b')$.

Exercise Check the new derived couple is an exact couple.

Clearly associated to an exact couple we have a spectral sequence. Determining converge will take a small amount of fiddling.

First In the case that we care about, $A^1_{p,q} = H_{p+q}(X_p)$

$$E^1_{p,q} = H_{p+q}(X_p, X_{p-1})$$

where we continue to have $X_p = \pi^{-1}(B^p)$ as a filtration of E.

More generally If C_* has an ascending filtration, we can take $A = \bigoplus_p F_p C_*$. The map $i : A \longrightarrow A$ is

$$\left.\begin{tikzcd}[row sep=small] i:\ F_0 \arrow[dr, hook] & F_0 \\ \oplus & \oplus \\ F_1 \arrow[dr, hook] & F_1 \\ \oplus & \oplus \\ F_2 & F_2 \end{tikzcd}\right\}$$

Just inclusion into the next filtration level.

Then we have
$$\begin{tikzcd} \bigoplus F_p C_* \arrow[rr, "i"] & & \bigoplus F_p C_* \arrow[dl] \\ & gr(C_*) = \operatorname{coker}(i) \arrow[ul, "\partial"] & \end{tikzcd}$$

Last Structure   Double complexes

We say

$$\begin{tikzcd}
C_{02} & C_{12} \arrow[l, "\delta_1"'] & C_{22} \arrow[l, "\delta_1"'] \arrow[d] \\
C_{01} \arrow[u, phantom] & C_{11} \arrow[l, "\delta_1"'] & C_{21} \arrow[l, "\delta_1"'] \arrow[d] \\
C_{00} & C_{10} \arrow[l, "\delta_1"'] & C_{20} \arrow[l, "\delta_1"']
\arrow[from=1-1, to=2-1, "\delta_0"]
\arrow[from=2-1, to=3-1, "\delta_0"]
\arrow[from=1-2, to=2-2, "\delta_0"]
\arrow[from=2-2, to=3-2, "\delta_0"]
\end{tikzcd}$$

is a <u>double complex</u> if $\delta_0^2 = 0$ and $\delta_1^2 = 0$.

We produce a <u>total complex</u> with differential

$$d_{p,q} = \delta_0 + (-1)^q \delta_1 .$$

This comes with a natural ascending filtration, of which the columns are the quotients. So there is a spectral sequence for which the E_1 page is homology with respect to $\delta_0 := d_0$.

$$\begin{tikzcd}
E^1_{02} & E^1_{12} \arrow[l, "\delta_1'"'] & E^1_{22} \arrow[l, "\delta_1'"'] \\
E^1_{01} & E^1_{11} \arrow[l, "\delta_1'"'] & E^1_{21} \arrow[l, "\delta_1'"'] \\
E^1_{00} & E^1_{10} \arrow[l, "\delta_1'"'] & E^1_{20} \arrow[l, "\delta_1'"']
\end{tikzcd}$$

Have an induced map

$\delta_1'[b_0] = 0 \iff$

$\delta_1 b_0 = \delta_0 b_1$ for some b_1.

$$\begin{tikzcd}
{} & b_1 \arrow[l, dashed] \arrow[d] & \\
 & \delta_1 b_0 & b_0 \arrow[l]
\end{tikzcd}$$

But what if?

We take homology with respect to $d_1 := \delta_1'$. Now what happens?

We can define $d_2[b_0] = \delta_1'[b_1]$ wherever $\delta_0 b_1 = \delta_1 b_1$.

This gives maps

$$\begin{tikzcd}
\delta_1 b_1 & b_1 \arrow[l] \arrow[d] \\
\delta_1 b_0 & b_0 \arrow[l]
\end{tikzcd}$$

in the original complex.

One continues to do this: The differential on the rth page corresponds to a staircase of related elements

$$
\begin{tikzcd}
\delta_2 b_2 & b_2 \arrow[l, "\delta_2"'] \arrow[d] & \\
& \delta_1 b_1 & b_1 \arrow[l, "\delta_1"'] \arrow[d, "\delta_0"] & \\
& & \delta_1 b_0 & b_0 \arrow[l, "\delta_1"']
\end{tikzcd}
$$

Exercise: Check this is well-defined.

<u>Example</u>

<u>E^0</u>

$$
\begin{tikzcd}
\mathbb{F}\langle a,b\rangle \arrow[d] & \mathbb{F}\langle a,b\rangle \arrow[l] \arrow[d] & \mathbb{F}\langle a,b\rangle \arrow[l] \arrow[d] \\
\mathbb{F}\langle c,d\rangle & \mathbb{F}\langle c,d\rangle \arrow[l] & \mathbb{F}\langle c,d\rangle \arrow[l]
\end{tikzcd}
$$

$$\delta_1 \begin{cases} a, b \mapsto a+b \\ c, d \mapsto c+d \end{cases}$$

$$\delta_0 \begin{cases} a, b \mapsto c+d \end{cases}$$

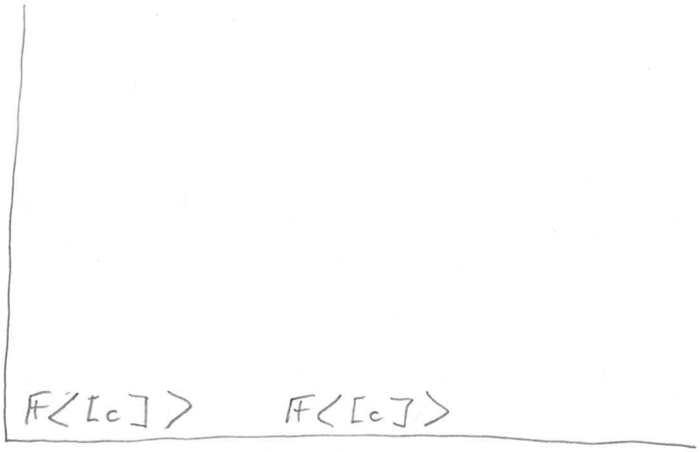
<u>E^1</u>

<u>E^2</u>

$$
\begin{tikzcd}
\mathbb{F}\langle [a+b]\rangle & \mathbb{F}\langle [a+b]\rangle & \mathbb{F}\langle [a+b]\rangle \\
\mathbb{F}\langle [c]\rangle & \mathbb{F}\langle [c]\rangle & \mathbb{F}\langle [c]\rangle \arrow[ull] & {} \arrow[ull]
\end{tikzcd}
$$

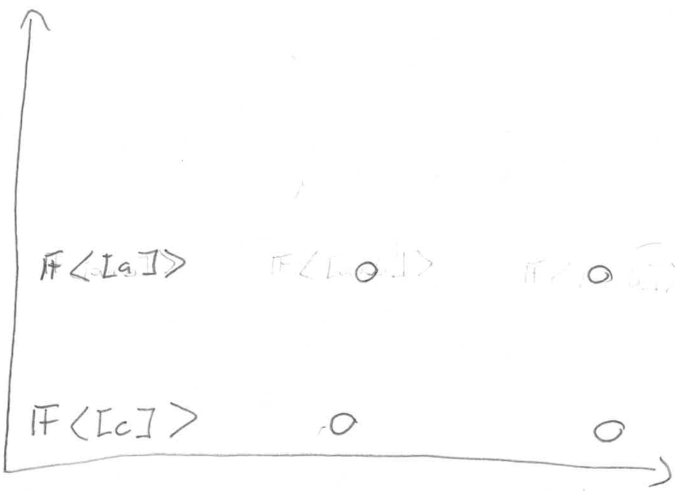
$$
\begin{tikzcd}
a+b & a \arrow[l] \arrow[d] \\
& c+d & c \arrow[l]
\end{tikzcd}
$$

E^3


$\mathbb{F}\langle[c]\rangle$
$\mathbb{F}\langle[c]\rangle$

OR

E^4


$\mathbb{F}\langle[a]\rangle$
$\mathbb{F}\langle[c]\rangle$
0
0
0
0