Lecture 12

Beginning Spectral Sequences

Motivating question: What, if anything, can we say about the homology of the spaces in a fibration \( F \longrightarrow E \rightarrow B \)?

General Remarks

1. Generalization of Exact Sequences
2. Arise naturally lots of places
   - Kunneth Theorem for nontrivial fibrations
     \( \Rightarrow \) Serre spectral sequence
   - Universal coefficient theorem for rings other than \( \mathbb{Z} \)
     \( \Rightarrow \) U.C. spectral sequence
   - Cellular cohomology for generalized (co)homology theories
     \( \Rightarrow \) Atiyah-Hirzebruch spectral sequence
   - Mayer-Vietoris for more than two sets
     \( \Rightarrow \) M-V spectral sequence
References

- Hatcher Spectral Sequences book
- Bott & Tu
- McCleary User's Guide to Spectral Sequences
- Chow "You could have invented Spectral Sequences"

Today chain complexes are over a field $k$.

**Def.** A (first quadrant) spectral sequence of homological type consists of $k$-vector spaces $E^n_{p,q}$ for $n \in \mathbb{N}, p,q \in \mathbb{Z}$ and maps

$$d^r_{p,q}: E^r_{p,q} \rightarrow E^r_{p-r, q+r-1}$$

such that

- $d^r_{p-r, q+r-1} \circ d^r_{p,q} = 0$

- $E^{r+1}_{p,q} = \ker(d^r_{p,q}) / \text{im}(d^r_{p-r, q+r-1})$

- $E^n_{p,q} = 0$ if $p < 0$ or $q < 0$ (first quadrant condition).

**Interpretation** This is book with a grid of vector spaces on each page. The number $r$ is the page number.
Page $r = 0$

- Note that $(P_{q})$ is just a Cartesian coordinate

- $d_{P_{q}}^{0} : E_{0}^{0} \rightarrow E_{P_{q-1}}^{0}$

Page $r = 1$

- $d_{P_{q}}^{1} : E_{1}^{1} \rightarrow E_{P_{q-1}}^{1}$
- Each $E_{P_{q-1}}^{1}$ is the homology of the previous page, e.g.

\[ E_{01}^{1} = \ker (d_{01}^{0}) / \text{im} (d_{22}^{0}) \]

Page $r = 2$

- $d_{P_{q}}^{2} : E_{2}^{2} \rightarrow E_{P_{q-2}, q+1}^{2}$
How to remember this? Think of the entire thing as graded along diagonals. Differentials always lower grading by 1.

**Defn.** The limit of a spectral sequence \( E^r_{p,q} \) is defined as follows: For each \((p, q) \in \mathbb{N} \times \mathbb{N} \)

\[
E_{p,q}^\infty = E_{p+1,q-1}^\infty = E_{p+2,q-2}^\infty = \ldots \quad \text{because of the first quadrant condition,}
\]

call this vector space \( E_{p,q}^\infty \). Let \( E_{n=0}^\infty = \bigoplus_{p+q=n} E_{p,q}^\infty \). Then the spectral sequence converges to \( \bigoplus_{n=0} E_n^\infty \).

**Defn.** A first quadrant spectral sequence of cohomological type is some \((E_{r}, \partial_{r}, d_{r}, p, q)\) s.t. \( d_{r} \circ \partial_{r} = 0 \), i.e., with same conditions as previously.

**Warning.** The \( r \)th page determines the modules on the \((r+1)\)st page, but not the differentials.
We're going to talk about how spectral sequences arise in general, but it's worth getting the first major example into play.

**Topology Application: Serre Spectral Sequence**

**Thm (Serre)** Let $F \to E$ be a Fibration. Assume $\pi_1(\emptyset)$ acts trivially on $H^*(F; k)$. (Often satisfied by just having $\emptyset$ be simply connected). Then there is a spectral sequence with $E^2_{p,q} = H^p(\emptyset; H^q(F; k)) \Rightarrow H^*_p(\emptyset; k) \otimes H^*_q(F; k)$ converging to $H^*_E(F; k)$.

This step only if $k$ is a field.

Something slightly more complicated is true for rings.

Similarly, if a spectral sequence with $E^2_{p,q} = H^p(\emptyset; H^q(F; k)) \Rightarrow H^*_p(\emptyset; k) \otimes H^*_q(F; k)$ converging to $H^*_E(F; k)$.

Compare to the **Kunneth Thm** $H^*_E(F \times \emptyset; k) \cong H^*_E(F; k) \otimes H^*_E(\emptyset; k)$.

This is the claim that the spectral sequence "collapses" (i.e., stops changing) at $E^2$.

Furthermore we see $H^*_E(Fiber \; Fibration)$ is no larger than $H^*_E(Corresponding \; Product)$. 
Example: $H^*(\mathbb{C}P^\infty)$

Path-loop fibration $S^1 \simeq \Omega \mathbb{C}P^\infty \rightarrow P\mathbb{C}P^\infty$

We know the s.s. converges to the homology of a contractible space.

With $k$ a field, the homology of $S^1$ terms just vanish in the tensor product, and this looks like:

$H^0(\mathbb{C}P^\infty) \rightarrow H_1(\mathbb{C}P^\infty) \rightarrow H_2(\mathbb{C}P^\infty)$
So we must have:
\[
\begin{align*}
H_0 (\Omega \mathbb{P}^\infty; 1k) &= 1k \\
H_1 (\Omega \mathbb{P}^\infty; 1k) &= 0 \\
H_2 (\Omega \mathbb{P}^\infty; 1k) &= 1k \\
\vdots
\end{align*}
\]

by examining the diagram.

Example 2: \( H_+ (\Omega \mathbb{S}^2) \)

\( \Omega \mathbb{S}^2 \longrightarrow \mathbb{P} \mathbb{S}^2 \)

\( S^2 \) is simply connected

\[ E^2\text{-page} \]

\[ \begin{array}{cccc}
0 & H_0 (\Omega \mathbb{S}^2) \otimes H_0 (S^2) & H_1 (\Omega \mathbb{S}^2) \otimes H_0 (S^2) & H_2 (\Omega \mathbb{S}^2) \\
0 & H_0 (\Omega \mathbb{S}^2) \otimes H_1 (S^2) & H_1 (\Omega \mathbb{S}^2) \otimes H_1 (S^2) & H_2 (\Omega \mathbb{S}^2) \\
0 & H_0 (\Omega \mathbb{S}^2) \otimes H_0 (S^2) & H_1 (\Omega \mathbb{S}^2) \otimes H_0 (S^2) & H_1 (\Omega \mathbb{S}^2) \otimes H_0 (S^2) \\
\end{array} \]
\[\begin{cases} 
H_0(\Omega S^2) = \mathbb{Z} \\
H_1(\Omega S^2) = 0 \\
H_2(\Omega S^2) = 0 \\
H_3(\Omega S^2) = \mathbb{Z} \\
H_{n+2}(\Omega S^2) = H_n(\Omega S^2) 
\end{cases}\]

**Exercise**  \( H_+ (\Omega S^n) \)

**Remark** This result is also accessible via the Morse theory we waved our hands at earlier.

**One more example** Fast proof of Hurewicz

**Thm** Let \( x \) be an \((n-1)\)-connected CW cpx, \( n \geq 2 \), then \( \pi_n (x) = H_n(x) \).

**PF** First assume we know the \( n=2 \) case.

**Lemma** IF \( x \) is \( m \)-connected then \( H_n(x) = 0 \) for \( 0 \leq m \leq n \).

**PF** A model for \( x \) w/ no cells in dimns \( \leq m+1 \) except for a single zero cell.

Now consider \( \Sigma x \rightarrow PX \)

\[\begin{array}{ccc}
\Sigma x & \rightarrow & PX \\
\downarrow & & \\
X & \leftarrow & \text{simply-connected}
\end{array}\]

We have a Serre spectral sequence

I want to do this one in \( \mathbb{Z} \)-coefficients.
Since $\Omega x$ is $(n-2)$ connected, get a large box of zeroes.

Preserved until the $E^n$ page, when we have

$$H_n(x) \xrightarrow{\partial} H_{n-1}(\Omega x),$$

So $d_n : H_n(x) \xrightarrow{\partial} H_{n-1}(\Omega x) \simeq \pi_{n-1}(\Omega x) \simeq \pi_n(x)$

Inductive assumption

For $n=2$ case: For $x$ ctd, $H_2(x) \cong \pi_2(x)/[x, x]$, but $\pi_1(\Omega x)$ is abelian.
Continuing this argument one also notes

\[ H_{n+t}(x) \cong H_{n+t-1}(\Omega^2 x) \text{ for } 0 \leq t \leq n-2. \]

Indeed, Serre's mod-c Hurewicz Thm: \textit{If } \( H_* (\omega) \) \textit{ is } \( 2^n \)-torsion then so is \( H_* (x) \) and vice versa.