Lecture 12

Beginning Spectral Sequences

Motivating question: What, if anything, can we say about the homology of the spaces in a fibration

$$\begin{array}{ccc} F & \hookrightarrow & E \\ & & \downarrow \\ & & B \end{array}$$

?

General Remarks

1. Generalization of Exact Sequences
2. Arise naturally lots of places
   - "Kunneth Theorem" for nontrivial fibrations
     $\rightsquigarrow$ Serre spectral sequence
   - Universal coefficient theorem for rings other than $\mathbb{Z}$
     $\rightsquigarrow$ u.c. spectral sequence
   - Cellular (co)homology for generalized (co)homology theories
     $\rightsquigarrow$ Atiyah-Hirzebruch spectral sequence
   - Mayer-Vietoris for more than two sets
     $\rightsquigarrow$ M-V spectral sequence

References

- Hatcher Spectral Sequences book
- Bott & Tu
- McCleary "User's Guide to Spectral Sequences"
- Chow "You Could Have Invented Spectral Sequences"

Today Chain complexes are over a field $\mathbb{k}$.

Defn A (first quadrant) spectral sequence of homological type consists of $\mathbb{k}$-vector spaces $E^r_{p,q}$ for $r \in \mathbb{N}$, $p,q \in \mathbb{Z}$ and maps $d^r_{p,q} : E^r_{p,q} \longrightarrow E^r_{p-r,\, q+r-1}$ st

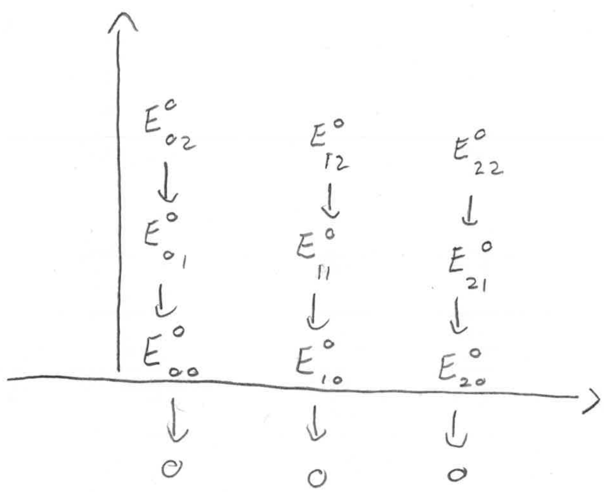
- $d^r_{p-r,\, q+r-1} \circ d^r_{p,q} = 0$

- $E^{r+1}_{p,q} = \ker(d^r_{p,q}) \big/ \operatorname{im}(d^r_{p+r,\, q-r+1})$

- $E^r_{p,q} = 0$ if $p<0$ or $q<0$ (first quadrant condition.

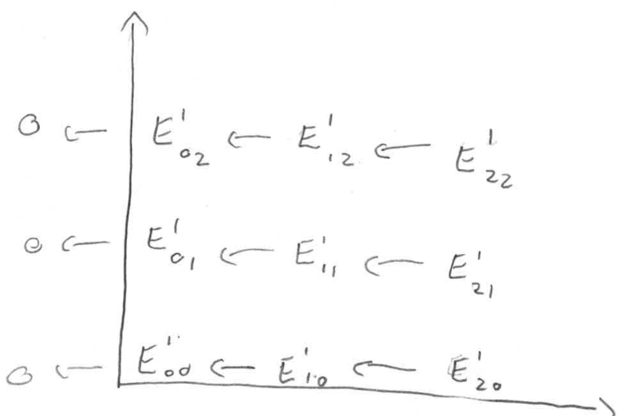
Interpretation This is book with a grid of vector spaces on each page. The number r is the page number.

Page $r=0$



- Note that (p,q) is just a Cartesian coordinate
- $d^0_{p,q} : E^0_{p,q} \longrightarrow E^0_{p,q-1}$

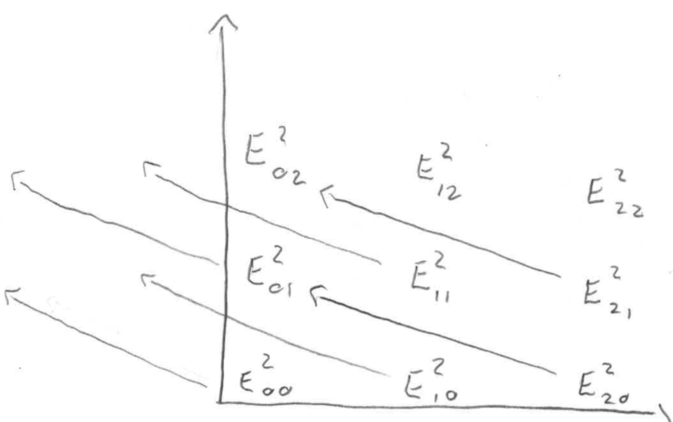
Page $r=1$



- $d^1_{p,q} : E^1_{p,q} \longrightarrow E^1_{p-1,q}$
- Each $E^1_{p,q}$ is the homology of the previous page, eg

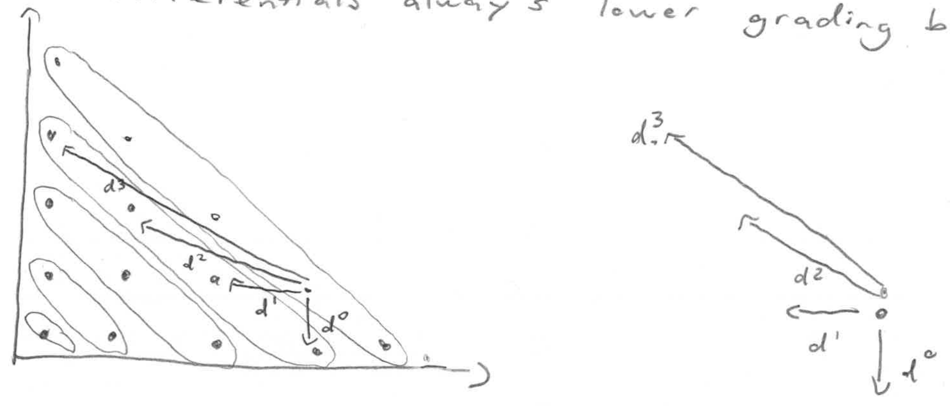
$$E^1_{0,1} = \ker(d^0_{0,1}) / \operatorname{im}(d^0_{0,2})$$

Page $r=2$



- $d^2_{p,q} : E^2_{p,q} \longrightarrow E^2_{p-2,\,q+1}$

How to remember this? Think of the entire thing as graded along diagonals. Differentials always lower grading by 1,



Defn The limit of a spectral sequence $\{E^r_{p,q}, d^r_{p,q}\}$ is defined as follows: For each (p,q) $\exists$ N st $E^N_{p,q} \simeq E^{N+1}_{p,q} \simeq E^{N+2}_{p,q} \simeq \dots$ because of the first quadrant condition.

Call this vector space $E^\infty_{p,q}$. Let $E^\infty_n = \bigoplus_{p+q=n} E^\infty_{p,q}$. Then the spectral sequence converges to $\bigoplus_{n=0}^{\infty} E^\infty_n$.

Defn A first quadrant spectral sequence of cohomological type is some $(E_r^{p,q}, d_r^{p,q})$ st $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,\, q-r+1}$ w/ same conditions as previously

Warning rth page determines the modules on the (r+1)st page, but not the differentials.

We're going to talk about how spectral sequences arise in general, but its worth getting the first major example into play.

Topology Application: Serre Spectral Sequence

Thm (Serre) Let $\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow \\ & & B \end{array}$ be a fibration. Assume $\pi_1(B)$ acts

trivially on $H_*(F;\mathbb{k})$. (Often satisfied by just having B be simply connected). Then there is a spectral sequence with

$$E^2_{p,q} = H_p(B; H_q(F;\mathbb{k})) \simeq H_p(B;\mathbb{k}) \otimes H_q(F;\mathbb{k}) \text{ converging}$$

This step only if $\mathbb{k}$ is a field

to $H_*(E;\mathbb{k})$. Something slightly more complicated is true for rings.

Similarly, $\exists$ a spectral sequence w/ $E_2^{p,q} = H^p(B; H^q(F;\mathbb{k}))$
$\simeq H^p(B;\mathbb{k}) \otimes H^q(F;\mathbb{k})$ converging to $H^*(E;\mathbb{k})$.

Compare to the Kunneth Thm $H_*(F\times B;\mathbb{k}) \simeq H_*(F;\mathbb{k}) \otimes H_*(B;\mathbb{k})$

This is the claim that the spectral sequence "collapses" (i.e., stops changing) at E^2.

Furthermore we see H_*(Fiber Bundle) is no larger than H_*(corresponding product).

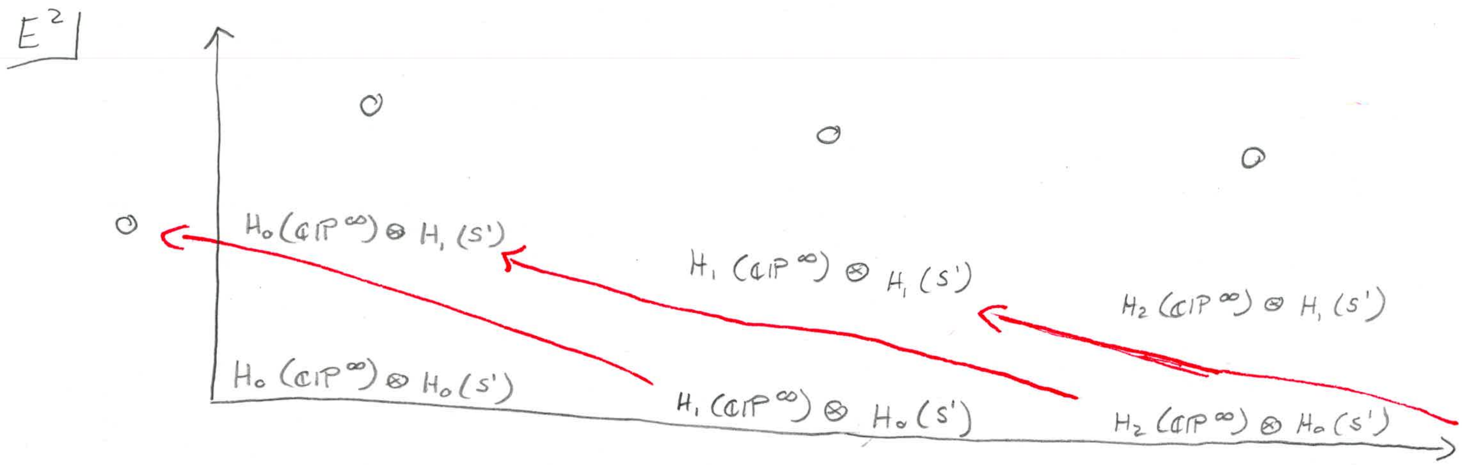
Example! $H^*(\mathbb{CP}^\infty)$

Path-loop fibration

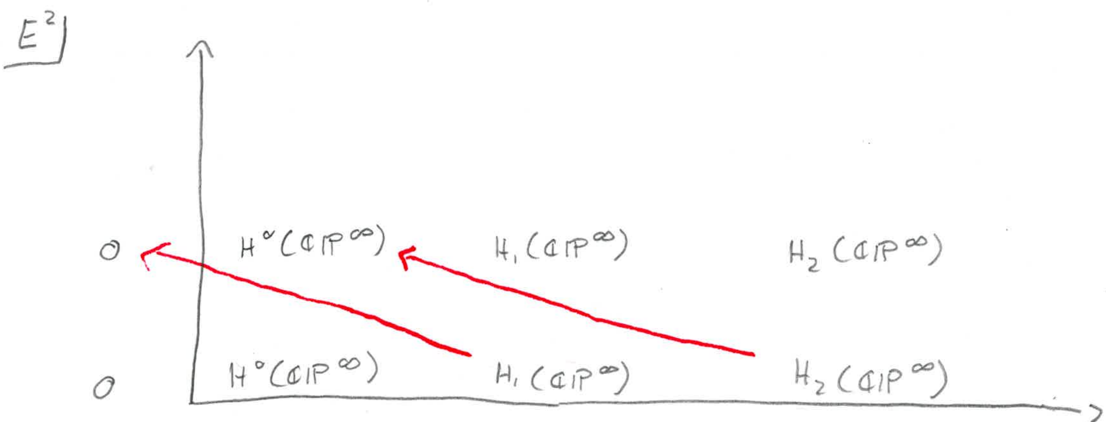
$$\begin{tikzcd} S^1 \simeq \Omega\mathbb{CP}^\infty \arrow[r] & P\mathbb{CP}^\infty \arrow[d] \\ & \mathbb{CP}^\infty \end{tikzcd}$$

← Note simply-connected base

We know the s.s. converges to the homology of a contractible space.



With $\mathbb{k}$ a field, the homology of S^1 terms just vanish in the tensor product, and this looks like.



Has to converge to $H_*(P\mathbb{CP}^\infty) = \left\{ \begin{array}{c} \vdots \\ 0 \\ 0 \\ 0 \\ \mathbb{k} \end{array} \right.$

So we must have $\begin{cases} H_0(\mathbb{CP}^\infty; \mathbb{k}) = \mathbb{k} \\ H_1(\mathbb{CP}^\infty; \mathbb{k}) = 0 \\ H_2(\mathbb{CP}^\infty; \mathbb{k}) = \mathbb{k} \\ \vdots \end{cases}$

by examining the diagram.

<u>Example 2</u> $H_*(\Omega S^2)$

$$\begin{tikzcd} \Omega S^2 \arrow[r] & PS^2 \arrow[d] \\ & S^2 \end{tikzcd}$$

S^2 ⟵ simply connected

<u>E^2-page</u>

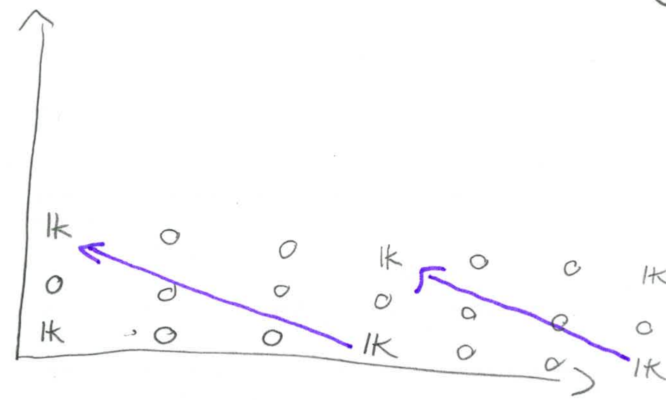
$$\begin{array}{c|ccc}
0 & H_0(\Omega S^2)\otimes H_2(S^2) & H_1(\Omega S^2)\otimes H_2(S^2) & H_1(\Omega S^2 \\
0 & H_0(\Omega S^2)\otimes H_1(S^2) & H_1(\Omega S^2)\otimes H_1(S^2) & H_1(\Omega S^2)\otimes H_1(S^2) \\
0 & H_0(\Omega S^2)\otimes H_0(S^2) & H_1(\Omega S^2)\otimes H_0(S^2) & H_1(\Omega S^2)\otimes H_0(S^2) \\
\hline
 & 0 & 0 & 0
\end{array}$$

E^2

E^3

$$\begin{tikzcd}
H_0(\Omega S^2) & H_1(\Omega S^2) & H_2(\Omega S^2) & H_3(\Omega S^2) \\
0 & 0 & 0 & 0 \\
H_0(\Omega S^2) & H_1(\Omega S^2) & H_2(\Omega S^2) \arrow[ull, red] & H_3(\Omega S^2) \arrow[ull, red] \arrow[uulll, violet]
\end{tikzcd}$$

$$\begin{cases} H_0(\Omega S^2) = \mathbb{k} \\ H_1(\Omega S^2) = 0 \\ H_2(\Omega S^2) = 0 \\ H_3(\Omega S^2) = \mathbb{k} \\ H_{n+3}(\Omega S^2) = H_n(\Omega S^2) \end{cases}$$



Exercise $H_*(\Omega S^n)$

Remark This result is also accessible via the Morse theory we waved our hands at earlier.

One more example Fast proof of Hurewicz

Thm Let X be an $(n-1)$-connected CW cpx, $n \geq 2$, then $\pi_n(X) = H_n(X)$,

PF First assume we know the $n=2$ case.

Lemma If X is m-connected then $H_i(X) = 0$ for $0 < i \leq m$.

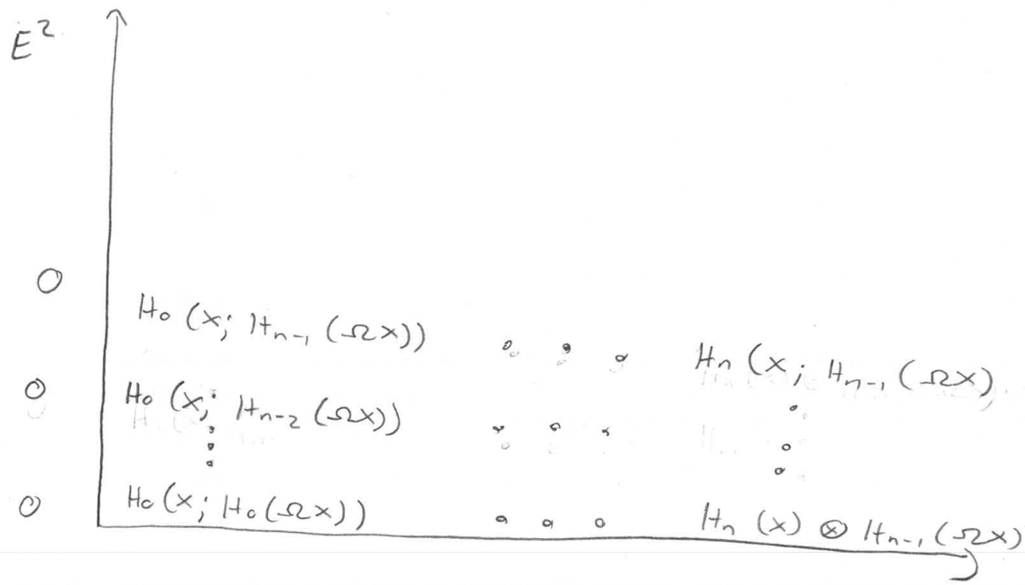
PF $\exists$ a model for X w/ no cells in dimns $\leq m+1$ except for a single zero cell.

Now consider

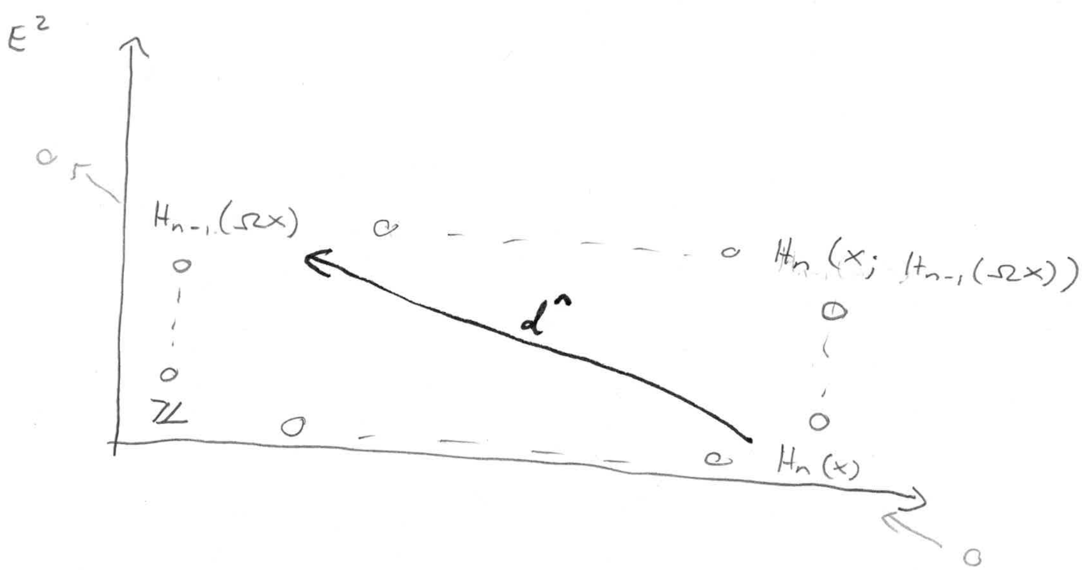
$$\begin{array}{ccc} \Omega X & \longrightarrow & PX \\ & & \downarrow \\ & & X \end{array} \leftarrow \text{simply-connected.}$$

We have a Serre spectral sequence

[want to do this one in $\mathbb{Z}$-coefficients]



Since ΩX is $(n-2)$ connected, get a large box of zeroes



Preserved until the E^n page, when we have

$H_n(X) \xrightarrow{\sim} H_{n-1}(\Omega X)$,

So $d_n : H_n(X) \xrightarrow{\sim} H_{n-1}(\Omega X) \cong \pi_{n-1}(\Omega X) \cong \pi_n(X)$

Inductive assumption

<u>$n=2$ case</u> For X ctd, $H_1(X) \cong \pi_1(X)/[X, X]$, but $\pi_1(\Omega X)$ is abelian.

Continuing this argument one also notes

$$H_{n+t}(X) \simeq H_{n+t-1}(\Omega X) \quad \text{For } 0 \le t \le n-2.$$

<u>Indeed</u> "Serre's mod-C Hurewicz Thm" If $H_*(X)$ is 2^n-torsion then so is $\pi_*(X)$ and vice versa.