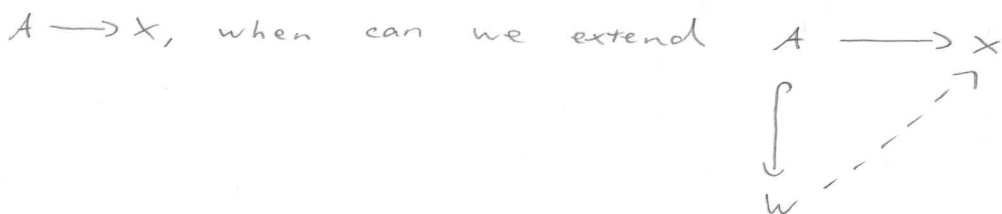


Beginning Obstruction Theory

Extension Problems Given a CW pair  $(W, A)$  and a map  $A \rightarrow X$ , when can we extend



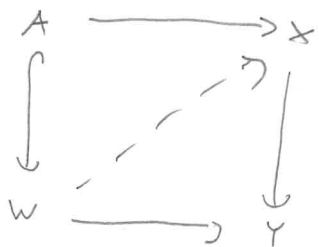
Lifting Problems Given a Fibration  $X \rightarrow Y$  and a map  $W \rightarrow Y$ ,

is there a map



Combine into the Relative Lifting Problem.

Given a CW pair  $(W, A)$ , a Fibration  $X \rightarrow Y$  and a map  $W \rightarrow Y$ , does there exist  $W \rightarrow X$  extending the lift on  $A$ ?



Obstruction Theory refers to the problem of defining a cohomology class of sequence of cohomology classes which vanish if and only if the problem has a solution.

Example What is the obstruction to extending

$$f: S^1 \rightarrow S^1 \text{ to } \mathbb{P}^2? \text{ Lies in } H^1(S^1; \mathbb{Z}) \cong H^2(\mathbb{P}^2, S^1; \pi, S^1)$$

What is the obstruction to extending a map  $f: X' \rightarrow S^1$  to  $X$ ? Lies in  $H^2(X, X'; \pi, S^1) \cong H^2(X, X'; \pi, S^1)$ .

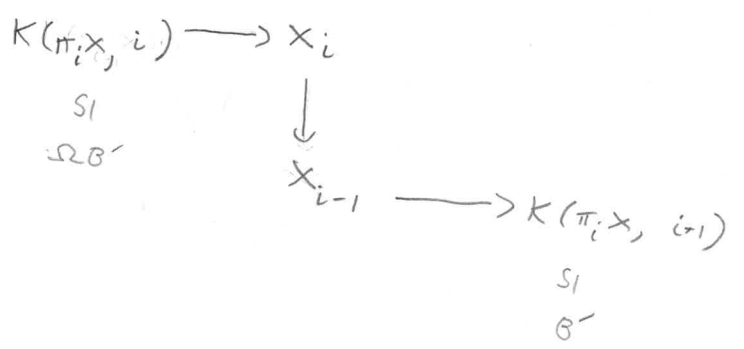
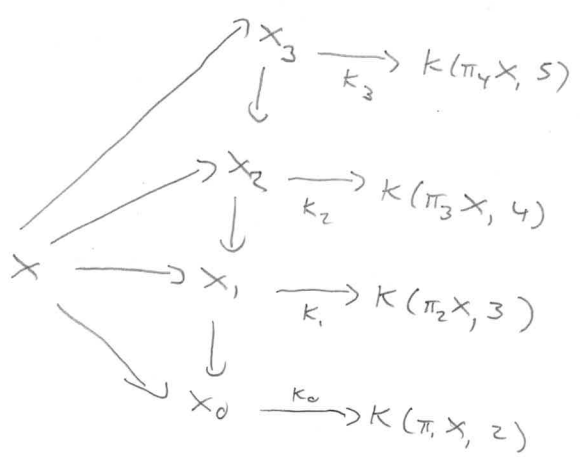


This perspective will come up more in relation to vector bundles.

Extension Problem

Suppose  $X$  has a Postnikov tower of principal fibrations.

Recall from last time: In the Postnikov tower for  $X$ , we have maps  $k_i$  extending the principal fibrations



Recall

$$X \rightarrow X_i \rightarrow \{ \mathbb{Z} \}$$

||

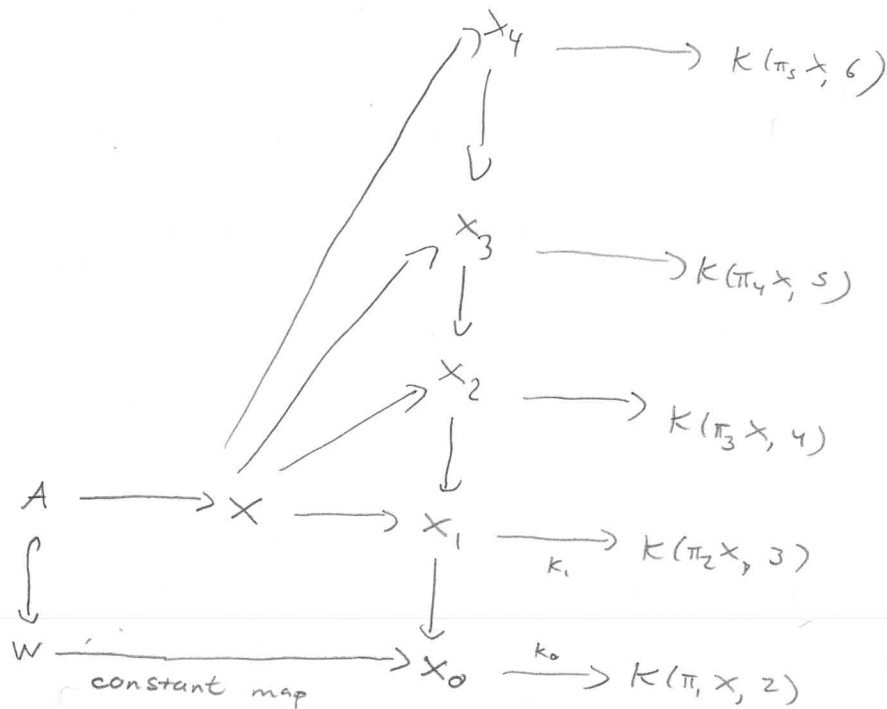
$\mathbb{Z}_{(i)}$

is an  $(i-1)$ -connected approximation to

$$X \rightarrow \{ \mathbb{P} \}$$

Note that including this step requires that  $\pi_i X$  be abelian

So we have



Idea: Lift  $W \rightarrow X_0$  to higher steps in the tower.

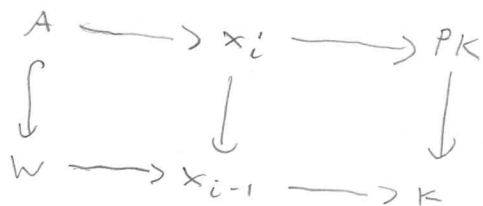
Notice that in the sequence of maps

$$K(\pi_i X, i) \rightarrow X_i \rightarrow X_{i-1} \xrightarrow{k_{i-1}} K(\pi_i X, i+1)$$

"   
 K

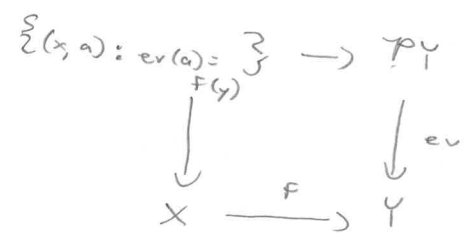
$X_i$  must be (up to homotopy) the homotopy fibre of  $k_{i-1}$ .

So we have



$$X_i = \{ (x, \gamma) : x \in X_{i-1}, \gamma : \mathbb{I} \rightarrow K, \gamma(0) = x, \gamma(1) = \text{basepoint} \}$$

[Another way of saying this: A principal fibration is always the pullback of a path fibration



So  $W \rightarrow X_n$  a lift is a nullhomotopy of the map  $W \rightarrow K$ . [A well defined path from every point in the image of  $W \rightarrow K$  to the basepoint in  $K$ .] We already have this lift on  $A$ , so really we start with a map of  $F$

$$W \cup CA \rightarrow K = K(\pi_n X, i+1)$$

This is a cohomology class  $w_n \in H^{n+1}(W \cup CA; \pi_n X) \cong H^{n+1}(W, A; \pi_n X)$ .

Lemma A lift  $W \rightarrow X_n$  extending  $A \rightarrow X_n$  exists  $\Leftrightarrow w_n = 0$ .

Proof The claim is that if  $w_n = 0$ , i.e.  $W \cup CA \rightarrow K$  is homotopic to a constant map, then we can extend to a map  $CW \rightarrow K$ . Let  $g_t: W \cup CA \rightarrow K$  be the nullhomotopy. Obviously the constant map on one side extends to  $cw$ , so by homotopy extension for  $(CW, W \cup CA)$ , we have a homotopy  $g_t: CW \rightarrow K$  ending in a constant map. Other direction is just restriction.

Now what? We need a way to relate extensions to  $X_n$  to  $X$ . That is, we need to be able to recover  $X$  from the tower.

Defn Let  $X_1 \xleftarrow{p_2} X_2 \xleftarrow{p_3} X_3 \xleftarrow{p_4} \dots$  be a sequence of maps. We say  $\varprojlim X_n = \{ (x_1, x_2, \dots) \in \prod X_n : p_i(x_i) = x_{i-1} \}$ .

Propn For the Postnikov tower of a connected CW complex  $X$ , the natural map  $X \rightarrow \varprojlim X_n$  is a weak htpy equivalence, so  $X$  is a CW approximation to  $\varprojlim X_n$ .

Follows From

Propn For a sequence of fibrations  $X_1 \leftarrow X_2 \leftarrow \dots$ , the natural map  $d: \pi_i(\varprojlim X_n) \rightarrow \varprojlim \pi_i(X_n)$  is surjective, and is injective if the maps  $\pi_i(X_n) \rightarrow \pi_i(X_{n-1})$  are surjective for  $n$  sufficiently large.

PF An element of  $\varprojlim \pi_i(X_n)$  is a set of maps  $F_n: (S^i, S_0) \rightarrow (X_n, x_n)$  w/  $[p_n \circ F_n] = [F_{n-1}]$  in  $\pi_i(X_{n-1})$ . By homotopy lifting and the fact that  $X_n \rightarrow X_{n-1}$  is a fibration, we can just insist  $p_n \circ F_n = F_{n-1}$ . [Do this inductively up the tower so we only change each map once.] This gives surjectivity (any element of  $\varprojlim \pi_i(X_n)$  corresponds to a set of maps into  $\varprojlim X_n$ , so comes from an element of  $\pi_i(\varprojlim X_n)$ ).

For injectivity, we can assume  $\pi_{i+1}(X_n) \rightarrow \pi_{i+1}(X_{n-1})$  is surjective for all  $n$ . Let  $F: S^i \rightarrow \varprojlim X_n$  and there are nullhomotopies  $F_n: D^{i+1} \rightarrow X_n$ . We have  $p_n F_n = F_{n-1}$  on  $S^i = \partial D^i$ , so together they form a map  $g_n: S^{i+1} \rightarrow X_{n-1}$ . But  $[g_{n-1}]$  is the image of some  $[g_n]$

w/  $g_n: S^{i+1} \rightarrow X_n$ . By, say, adjoining a sphere representing the class  $[g_n]$

to  $F_n$ , we can change to  $F'_n$  so the replacement map  $g_{n-1}$  is nullhomotopic, i.e.  $p_n F_n \cong F_{n-1}$  rel  $S^i$ . Applying homotopy lifting for  $(\mathcal{D}^{i+1}, S^i)$  we can make  $p F_n = F_{n-1}$ . Iterating this up the tower shows  $F: S^i \rightarrow \varprojlim X_n$  is nullhomotopic, so  $\iota$  is injective.

The original proposition now follows from

$$\underbrace{\pi_i(X) \xrightarrow{\quad} \pi_i(\varprojlim X_n)}_{\text{iso for large } n} \xrightarrow{\quad} \underbrace{\varprojlim \pi_i(X_n)}_{\text{iso since } \pi_{i+1}(X_n) \twoheadrightarrow \pi_{i+1}(X_{n-1}) \text{ for large } n}$$

Back to extension problem

Say we have extended  $A \rightarrow X_n$  to  $W \rightarrow X_n$  for all  $n$ , giving  $W \rightarrow \varprojlim X_n$ . extending  $A \rightarrow X \rightarrow \varprojlim X_n$ . Let  $M$  be the mapping cylinder of  $X \rightarrow \varprojlim X_n$ . The map  $F: W \rightarrow \varprojlim X_n \subseteq M$  has the property that  $F|_A$  factors through  $A$ , so there is a homotopy <sup>between</sup>  $F|_A$  and a map  $A \rightarrow X \subseteq M$ . This homotopy can be extended to  $W$  to get a map  $(W, A) \rightarrow (M, X)$ . Furthermore  $X \rightarrow \varprojlim X_n$  is a weak htpy equivalence, so by the compression lemma we can homotop to a map  $W \rightarrow X$  extending  $A \rightarrow X$  as desired.

Corollary IF  $X$  is a connected abelian CW complex and  $(W, A)$  is a CW pair such that  $H^{n+1}(W, A; \pi_n X) = 0$  for all  $n$ , then every map  $A \rightarrow X$  can be extended to a map  $W \rightarrow X$ ,

This can be used to extend the homology version of Whitehead's Theorem.

Propn IF  $X$  and  $Y$  are connected "abelian" CW cpxes, then a map  $f: X \rightarrow Y$  inducing isomorphisms on all homology groups is a homotopy equivalence.

PF Assume  $F$  is an inclusion. We want to extend the map  $X \xrightarrow{id} X$  to a retraction  $Y \rightarrow X$ . Since  $X$  is abelian, we can use obstruction theory. Since  $H_*(X) \xrightarrow{\sim} H_*(Y)$ , we have  $H_*(Y, X) = 0$ , so  $H^{n+1}(Y, X; \pi_n X) = 0 \forall n$ . So there are no obstructions and a retraction  $Y \rightarrow X$  exists

Ergo the maps  $\pi_n(Y) \rightarrow \pi_n(Y, X)$  are onto, so since  $\pi_1(X)$  acts trivially on  $\pi_n(Y)$ ,  $\pi_1(X)$  acts trivially on  $\pi_n(Y, X)$ . Then the relative Hurewicz thm implies  $\pi_n(Y, X) = 0$  for all  $n$ , so  $X \rightarrow Y$  is a weak hty equivalence.

One can do this for the relative lifting problem, but one only gets a sequence of classes whose vanishing implies the existence of a lift (not the other way around).