Lecture 11

Beginning Obstruction Theory

Extension Problems Given a CW pair (W, A) and a map $A \to X$, when can we extend

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & X \\ W \arrow[ur, dashed] & \end{tikzcd}$$

Lifting Problems Given a Fibration $X \to Y$ and a map $W \to Y$, is there a map

$$\begin{tikzcd} & X \arrow[d] \\ W \arrow[ur, dashed] \arrow[r] & Y \end{tikzcd}$$

Combine into the Relative Lifting Problem.

Given a CW pair (W, A), a Fibration $X \to Y$ and a map $W \to Y$, does there exist $W \to X$ extending the lift on A?

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & X \arrow[d] \\ W \arrow[ur, dashed] \arrow[r] & Y \end{tikzcd}$$

Obstruction Theory refers to the problem of defining a cohomology class of sequence of cohomology classes which vanish if and only if the problem has a solution.

<u>Example</u> What is the obstruction to extending $F: S^1 \longrightarrow S^1$ to D^2? Lies in $H^1(S^1; \mathbb{Z}) \simeq H^2(D^2, S^1; \pi_1 S^1)$

What is the obstruction to extending a map $F: X^1 \longrightarrow S^1$ to X? Lies in $H^2(X^2, X^1; \pi_1 S^1) \simeq H^2(X, X^1; \pi_1 S^1)$.

$\uparrow$

This perspective will come up more in relation to vector bundles.

<u>Extension Problem</u> Suppose X has a Postnikov tower of principal fibrations.

Recall from last time: In the Postnikov tower for X, we have maps k_i extending the principal fibrations

$$\begin{tikzcd}
 & X_3 \arrow[r, "k_3"] \arrow[d] & K(\pi_4 X, 5) \\
 & X_2 \arrow[r, "k_2"] \arrow[d] & K(\pi_3 X, 4) \\
X \arrow[ruu] \arrow[ru] \arrow[r] \arrow[rd] & X_1 \arrow[r, "k_1"] \arrow[d] & K(\pi_2 X, 3) \\
 & X_0 \arrow[r, "k_0"] & K(\pi_1 X, 2)
\end{tikzcd}$$

$$\begin{tikzcd}
K(\pi_i X, i) \arrow[r] & X_i \arrow[d] & \\
 & X_{i-1} \arrow[r] & K(\pi_i X, i+1)
\end{tikzcd}$$

$K(\pi_i X, i) \cong \Omega B'$, $\quad K(\pi_i X, i+1) \cong B'$

Note that including this step requires that $\pi_1 X$ be abelian

Recall

$X \to X_i \to \{x_0\}$, $\quad X_i = Z_{i+1}$

is an $(i+1)$-connected approximation to $X \to \{pt\}$

So we have

$$\begin{tikzcd}
 & & X_4 \arrow[r] \arrow[d] & K(\pi_5 X, 6) \\
 & & X_3 \arrow[r] \arrow[d] & K(\pi_4 X, 5) \\
 & & X_2 \arrow[r] \arrow[d] & K(\pi_3 X, 4) \\
A \arrow[r] \arrow[d, hook] & X \arrow[r] \arrow[ur] \arrow[uur] \arrow[uuur] & X_1 \arrow[r, "k_1"'] \arrow[d] & K(\pi_2 X, 3) \\
W \arrow[rr, "\text{constant map}"'] & & X_0 \arrow[r, "k_0"] & K(\pi_1 X, 2)
\end{tikzcd}$$

Idea: Lift $W \to X_0$ to higher steps in the tower.

Notice that in the sequence of maps

$$K(\pi_i X, i) \longrightarrow X_i \longrightarrow X_{i-1} \xrightarrow{k_{i-1}} \underset{\displaystyle K}{\overset{}{K(\pi_i X, i+1)}}$$

$$K(\pi_i X, i+1) = K$$

X_i must be (up to homotopy) the homotopy fibre of k_{i-1}.

So we have

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook] & X_i \arrow[r] \arrow[d] & PK \arrow[d] \\
W \arrow[r] & X_{i-1} \arrow[r] & K
\end{tikzcd}$$

$X_i = \{(x, \gamma) : x \in X_{i-1}, \ \gamma : I \to K, \ \gamma(0) = x, \ \gamma(1) = \text{basepoint}\}$

[Another way of saying this: A principal fibration is always the pullback of a path fibration

$$\begin{tikzcd}
\{(x, a) : ev(a) = f(y)\} \arrow[r] \arrow[d] & PY \arrow[d, "ev"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

So $W \to X_n$ a lift is a nulhomotopy of the map $W \to K$. [A well defined path from every point in the image of $W \to K$ to the basepoint in K.] We already have this lift on A, so really we start with a map of

$$W \cup CA \longrightarrow K = K(\pi_i X, i+1)$$

This is a cohomology class $w_n \in H^{n+1}(W \cup CA; \pi_n X) \simeq H^{n+1}(W, A; \pi_n X)$.

<u>Lemma</u> A lift $W \to X_n$ extending $A \to X_n$ exists $\Leftrightarrow$ $w_n = 0$.

<u>Proof</u> The claim is that if $w_n = 0$, i.e. $W \cup CA \to K$ is homotopic to a constant map, then we can extend to a map $CW \to K$. Let $g_t : W \cup CA \to K$ be the nulhomotopy. Obviously the constant map on one side extends to CW, so by homotopy extension for $(CW, W \cup CA)$, we have a homotopy $g_t : CW \to K$ ending in a constant map. Other direction is just restriction.

<u>Now what?</u> We need a way to relate extensions to X_n to X. That is, we need to be able to recover X from the tower.

<u>Defn</u> Let $X_1 \xleftarrow{p_2} X_2 \xleftarrow{p_3} X_3 \xleftarrow{p_4} \cdots$ be a sequence of maps. We say

$$\varprojlim X_n = \{ (x_1, x_2, \ldots) \in \prod X_n : p_i(x_i) = x_{i-1} \}.$$

Propn For the Postnikov tower of a connected CW cpx X, the natural map $X \to \varprojlim X_n$ is a weak htpy equivalence, so X is a CW approximation to $\varprojlim X_n$.

Follows from

Propn For a sequence of fibrations $X_1 \leftarrow X_2 \leftarrow \cdots$, the natural map $\lambda : \pi_i(\varprojlim X_n) \to \varprojlim \pi_i(X_n)$ is surjective, and is injective if the maps $\pi_{i+1}(X_n) \to \pi_{i+1}(X_{n-1})$ are surjective for n sufficiently large.

Pf An element of $\varprojlim \pi_i(X_n)$ is a set of maps $f_n : (S^i, s_0) \to (X_n, x_n)$ w/ $[p_n \circ f_n] = [f_{n-1}]$ in $\pi_i(X_{n-1})$. By homotopy lifting and the fact that $X_n \to X_{n-1}$ is a fibration, we can just insist $p_n \circ f_n = f_{n-1}$. [Do this inductively up the tower so we only change each map once.] This gives surjectivity/ any element of $\varprojlim \pi_i(X_n)$ corresponds to a set of maps into $\varprojlim X_n$, so comes from an element of $\pi_i(\varprojlim X_n)$.

For injectivity, we can assume $\pi_{i+1}(X_n) \to \pi_{i+1}(X_{n-1})$ is surjective for all n. Let $f : S^i \to \varprojlim X_n$ st there are nulhomotopies $F_n : D^{i+1} \to X_n$. We have $p_n F_n = F_{n-1}$ on $S^i = \partial D^i$, so together the form a map $g_{n-1} : S^{i+1} \to X_{n-1}$. But $[g_{n-1}]$ is the image of some $[g_n]$ w/ $g_n : S^{i+1} \to X_n$. By, say, adjoining a sphere representing the class $-[g_n]$

to F_n, we can change to F_n' so the replacement map g_{n-1} is nulhomotopic, i.e. $p_n F_n \simeq F_{n-1}$ rel S^i. Applying homotopy lifting for (D^{i+1}, S^i) we can make $pF_n = F_{n-1}$. Iterating this up the tower shows $F: S^i \to \varprojlim X_n$ is nulhomotopic, so λ is injective.

The original proposition now follows from

$$\underbrace{\pi_i(X) \longrightarrow \pi_i(\varprojlim X_n)}_{\text{iso for large } n} \xrightarrow{\lambda} \varprojlim \pi_i(X_n)$$

iso since $\pi_{i+1}(X_n) \twoheadrightarrow \pi_{i+1}(X_{n-1})$ for large n

<u>Back to extension problem</u>

Say we have extended $A \to X_n$ to $W \to X_n$ for all n, giving $W \to \varprojlim X_n$. extending $A \to X \to \varprojlim X_n$. Let M be the mapping cylinder of $X \to \varprojlim X_n$. The map $F: W \to \varprojlim X_n \subseteq M$ has the property that $F|_A$ factors through A, so there is a homotopy between $F|_A$ and a map $A \to X \subseteq M$. This homotopy can be extended to W to get a map $(W, A) \to (M, X)$. Furthermore $X \to \varprojlim X_n$ is a weak htpy equivalence, so by the compression lemma we can homotop to a map $W \to X$ extending $A \to X$ as desired.

<u>Corollary</u> If X is a connected abelian CW complex and (W, A) is a CW pair such that $H^{n+1}(W, A; \pi_n X) = 0$ for all n, then every map $A \to X$ can be extended to a map $W \to X$.

This can be used to extend the homology version of Whitehead's Theorem.

<u>Propn</u> If X and Y are connected "abelian" CW cpxes, then a map $f: X \to Y$ inducing isomorphisms on all homology groups is a homotopy equivalence.

<u>Pf</u> Assume f is an inclusion. We want to extend the map $X \xrightarrow{id} X$ to a retraction $Y \to X$. Since X is abelian, we can use obstruction theory. Since $H_*(X) \xrightarrow{\sim} H_*(Y)$, we have $H_*(Y, X) = 0$, so $H^{n+1}(Y, X; \pi_n X) = 0$ $\forall$ n. So there are no obstructions and a retraction $Y \to X$ exists. Ergo the maps $\pi_n(Y) \to \pi_n(Y, X)$ are onto, so since $\pi_1(X)$ acts trivially on $\pi_n(Y)$, $\pi_1(X)$ acts trivially on $\pi_n(Y, X)$. Then the relative Hurewicz thm implies $\pi_n(Y, X) = 0$ for all n, so $X \to Y$ is a weak htpy equivalence.

One can do this for the relative lifting problem, but one only gets a sequence of classes whose vanishing implies the existence of a lift (not the other way around).