Lecture 10

Last Time (Z, A) is an n-connected approximation to (X, A) if there is a map $F : (Z, A) \longrightarrow (X, A)$ inducing

$$\begin{cases} \pi_i(Z) \xrightarrow{\sim} \pi_i(X) & \text{For } i > n \\ \pi_i(A) \xrightarrow{\sim} \pi_i(Z) & \text{For } i < n \\ \pi_n(A) \twoheadrightarrow \pi_n(Z) \hookrightarrow \pi_n(X) \end{cases}$$

Similarly we can do this for a map $X \xrightarrow{F} Y$, interposing Z_n st

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & Z_n \arrow[r, "h"] & Y \arrow[d] \\ X \arrow[rr, "F"] & & Y \end{tikzcd}$$

$\ni$
$h_* : \pi_i(Z) \xrightarrow{\sim} \pi_i(Y)$ $i > n$
$g_* : \pi_i(X) \xrightarrow{\sim} \pi_i(Z)$ $i < n$
$\pi_n(X) \twoheadrightarrow \pi_n(Z) \hookrightarrow \pi_n(Y)$

Lemma If $n \geq n'$ and $\{Z_n, Z_{n'}\}$ are $\{n, n'\}$ connected approximations to F, we have a map

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Z_n \arrow[r] \arrow[d] & Y \\ & Z_{n'} \arrow[ur] & \end{tikzcd}$$

commuting up to homotopy.

More generally Given an n-connected model $F : (Z, A) \to (X, A)$, an n'-connected CW-model $F' : (Z', A') \longrightarrow (X', A')$, and a map $g : (X, A) \to (X', A')$

Then there is a map h st

$$\begin{tikzcd} Z \arrow[r, "F"] \arrow[d, "h"] & X \arrow[d, "g"] \\ Z' \arrow[r, "F'"] & X' \end{tikzcd}$$

commutes up to homotopy and $h|_A = g$.

PF After possibly replacing with a htpc CW cpx, we can assume $Z-A$ has no cells of dimn $\leq n$. Consider the relative mapping cylinder W' of F' obtained by collapsing each line segment $\{a'\} \times I$ to a point, for $a' \in A'$. This contains a copy of Z' and def retracts onto X', and the relative groups $\pi_i(W, Z')$ are zero for $i > n'$. [The point is to ensure that $W-Z'$ does not have cells of dimn $\leq n$.]

Now we can map $Z \xrightarrow{F} X \xrightarrow{g} X' \hookrightarrow W'$. By the compression lemma and $n' \geq n$, gF is homotopic rel A to some h w/ image in Z'.

Similarly, if we have two maps h_0 and h_1 st $h_0, h_1 : Z \to Z'$ and $F'h_0$, $F'h_1$ are both homotopic to gF rel A. Then as maps to W', h_0 and h_1 are homotopic rel A. By the compression lemma again we deform the homotopy $h_0 \simeq h_1$ rel A. $\square$

This led us to

Whitehead towers

$$
\begin{tikzcd}
pt \arrow[r] \arrow[dr] \arrow[ddr] & Z_0 \arrow[r] & X \\
& Z_1 \arrow[u] \arrow[ur] & \\
& Z_2 \arrow[u] \arrow[uur] &
\end{tikzcd}
$$

Postnikov towers

$$
\begin{tikzcd}
& Z_2 \arrow[d] \arrow[ddr] & \\
& Z_1 \arrow[d] \arrow[dr] & \\
X \arrow[r] \arrow[ur] \arrow[uur] & Z_0 \arrow[r] & pt
\end{tikzcd}
$$

For Postnikov towers, a more usual notation is

$$\begin{tikzcd} & & X_3 \arrow[d] \\ & & X_2 \arrow[d] \\ X \arrow[rr] \arrow[urr] \arrow[uurr] & & X_1 \end{tikzcd}$$

So this is $K(\pi_1 X, 1)$

- $X_K = Z_{K+1}$
- Note omission of $X_0 = \{pt\}$

Lemma $X_{n+1} \longrightarrow X_n$ can be taken to be a fibration w/ fiber $K(\pi_n(X), n)$.

eg.

$$\begin{tikzcd} K(\pi_2, 2) \arrow[r] & X_2 \arrow[d] \\ & K(\pi_1, 1) = X_1 \end{tikzcd}$$

Pf Replace whatever spaces and map we originally had w/ a fibration via the usual methods. (Do this one step at a time so that you change x_i and then use it as the target for the next fibration.) Then we have

$$\begin{tikzcd} \pi_n(F) \arrow[r] & \pi_n(X_n) \arrow[r] & \underset{\overset{\parallel}{0}}{\pi_n(X_{n-1})} \arrow[dll] \\ \pi_{n-1}(F) \arrow[r] & \pi_{n-1}(X_n) \arrow[r, "\sim"] & \pi_{n-1}(X_{n-1}) \end{tikzcd}$$

we see $\pi_n(F) \simeq \pi_n(X_n)$, $\pi_{n-1}(F) = 0$, $\pi_K(F) \simeq 0$ for $k < n$, $\pi_K(F) \simeq 0$ for $k > n$. So F is a $K(\pi_n(X), n)$.

Indeed, we can say a bit more.

__Defn__ A fibration is __principal__ if $\exists$ $F' \to E' \to B'$ st

we have weak htpy equivalences

$$\begin{tikzcd}
\Omega E \arrow[r] & \Omega B \arrow[r] & F \arrow[r] \arrow[d, "\wr"] & E \arrow[r] \arrow[d, "\wr"] & B \arrow[d, "\wr"] & \\
 & & \Omega B' \arrow[r] & F' \arrow[r] & E' \arrow[r] & B'
\end{tikzcd}$$

Suppose $K(\pi_{n+1}, n+1) \to X_{n+1} \to X_n$ is principal, then

$B \simeq \Omega K(\pi_{n+1}, n+1) \simeq K(\pi_{n+1}, n+2)$. Results in

$$\begin{tikzcd}
K(\pi_{n+1}, n+1) \arrow[r] & X_{n+1} \arrow[d] & \\
 & X_n \arrow[r, "k_n"] & K(\pi_{n+1}, n+2)
\end{tikzcd}$$

Notice this is an element of $H^{n+2}(X; \pi_{n+1})$.

This is the nth k-invariant.

In fact one can show these cohomology classes determine the tower!

__Thm__ If X is "abelian" ($\pi_1(X)$ acts trivially on $\pi_n(X)$ for all $n > 1$)

then all fibrations in the Postnikov tower are principal.

Indeed, you can think of this as a special case of a general question: When is a map equivalent to the inclusion of a fiber in a total space?

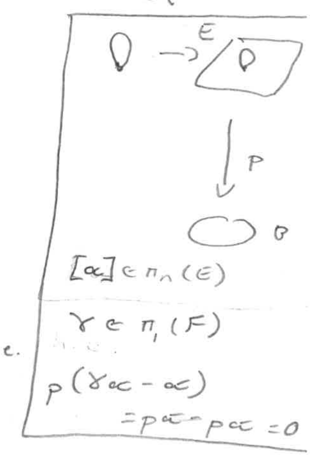
$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "s"'] & X \arrow[d] \\ F \arrow[r, hook] & E \end{tikzcd}$$

$\leftarrow$ Can assume $A \hookrightarrow X$ by the usual trick

So that would mean we would have

$$\begin{tikzcd} & E_i \arrow[d] \arrow[dr, dashed, no head] & \\ F_i \arrow[ur, dashed] \arrow[r, hook] \arrow[d, "\wr"] & A \arrow[r, hook, "i"] \arrow[d, "\wr"] & X \arrow[d, "\wr"] & \\ \Omega B \arrow[r] & F \arrow[r, hook] & E \arrow[r] & B \end{tikzcd}$$

Must have $F_i \simeq \Omega B \rightsquigarrow \pi_2(B) = \pi_1(F_i) = \pi_2(X, A)$ is an abelian group. F_i also comes w/ the structure of an H-space. And $\pi_1(A) = \pi_1(F)$ acts trivially on $\pi_n(X, A) = \pi_n(F_i) = \pi_{n-1}(B)$.



This is sufficient. Formally:

<u>Lemma</u> A CW inclusion $A \hookrightarrow X$ w/ fiber a $K(G, n)$ is h.e. to an $F \hookrightarrow E$, $E \to B$ $\Leftrightarrow$ $\pi_1(A)$ acts trivially on $\pi_n(X, A)$.

<u>Pf</u> Must have $B = K(G, n+1) = \Omega K(G, n)$. We have $\pi_i(X, A) = \begin{cases} 0 & i \neq n, \\ G & i = n \end{cases}$

So $H_n(X, A) \cong \pi_n(X, A) \cong G$ since $\pi_1(A)$ acts trivially on $\pi_n(X, A)$

But $G \simeq H_{n+1}(X/A) \simeq \pi_{n+1}(X/A)$. On X/A we have a canonical element of $H^{n+1}(X/A; G) \rightsquigarrow$ map $X/A \longrightarrow K(G, n+1)$.

So in total we have

$$\begin{tikzcd}
A \arrow[r] & X \arrow[r] \arrow[d] \arrow[rr, bend left, "f"] & X/A \arrow[r] \arrow[d, "s"] & K(G, n+1) \\
& F_f \arrow[r] & E_f \arrow[ur] &
\end{tikzcd}$$

Then $(X, A) \simeq (E_f, F_f)$ because $X/A \longrightarrow K(G, n+1)$ is an isomorphism on π_{n+1}, so can construct X/A (up to h.e.) by attaching cells of dimn $\geq n+3$ to $K(G, n+1)$.

<u>Proof of the thm</u> Have

$$\begin{tikzcd}
K(\pi_{n+1}, n+1) \arrow[r] & X_{n+1} \arrow[d] & \\
& X_n \arrow[r, "?"] & K(\pi_{n+1}, n+2)
\end{tikzcd}$$

<u>Need</u> $\pi_1(X_{n+1})$ acts trivially on $\pi_{n+2}(X_n, X_{n+1}) \iff \pi_1(X_{n+1})$ acts trivially on $\pi_{n+1}(K(\pi_{n+1}), n+1) \simeq \pi_{n+1}(X)$.

<u>Similarly</u> Every map $f: X \to Y$ between connected CW complexes has a Moore-Postnikov tower, which is unique up to homotopy equivalence. If $\pi_1(X)$ acts trivially on $\pi_n(M_f, X)$ for all $n > 1$, then this can be taken to be a tower of principal fibrations.

Relationship between Whitehead & Postnikov towers.

Recall that the Whitehead towers are towers of n-connected covers.

$$\begin{tikzcd}
\vdots \arrow[d] & \\
Z_2 \arrow[r] \arrow[d] & K(\pi_3 Y, 3) \\
Z_1 \arrow[r] \arrow[d] & K(\pi_2 Y, 2) \\
Y \arrow[r] & K(\pi_1 Y, 1)
\end{tikzcd}$$

One can take Z_n to be the homotopy fibre of the map Z_{n-1} which is the first nontrivial stage in a <u>Postnikov</u> tower for Z_{n-1}.

<u>Beginning Obstruction Theory</u>

<u>Extension Problems</u> Given a CW pair (W, A) and a map $A \to X$, when can we extend

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook] & X \\
W \arrow[ur, dashed] &
\end{tikzcd}$$

<u>Lifting Problems</u> Given a fibration $X \to Y$ and a map $W \to Y$, is

there a map
$$\begin{tikzcd} & X \arrow[d] \\ W \arrow[ur, dashed] \arrow[r] & Y \end{tikzcd}$$

Combine into the <u>Relative Lifting Problem</u>

Given a CW pair (W, A), a Fibration $X \to Y$, and a map $W \to Y$ does there exist $W \to X$ extending the lift on A?

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & X \arrow[d] \\ W \arrow[ur, dashed] \arrow[r] & Y \end{tikzcd}$$

<u>Obstruction theory</u> refers to the problem of defining a cohomology class or sequence of cohomology classes which vanish if and only if the problem has a solution.

<u>Example</u> What is the obstruction to extending a map $f: S^1 \to S^1$ to D^2? Lies in $H^1(S^1; \mathbb{Z}) \simeq H^2(D^2, S^1; \pi_1 S^1)$

What is the obstruction to extending a map

$f: X^1 \hookrightarrow S^1$ to X? Lies in $H^2(X^2, X^1; \pi_1 S^1)$

$$\begin{gathered} H^2(X^2, X^1; \pi_1 S^1) \\ \cong \\ H^2(X, X^1; \pi_1 S^1) \end{gathered}$$