Last semester you introduced the homotopy groups
\[ \pi_n(X, x_0) = \pi_n(S^n, s) \to (x, x_0) \text{ homotopy} \]

**Why care?**
- **Natural, easily defined**
- **Powerful**

**e.g. Whitehead's Theorem**
- If \( X, Y \) are CW complexes and \( F: X \to Y \) is a map such that \( \pi_n(X, x_0) \to \pi_n(Y, y_0) \) is an isomorphism for all \( n \), then \( F \) is a homotopy equivalence.

**Hurewicz**
- If \( X \) is a CW complex and \( \pi_n(X, x_0) \) is trivial for \( n < n \), where \( n > 1 \), then \( \pi_n(X) = 0 \) for \( i < n \), and \( \pi_n(X) = H_n(X) \).

"But hard to compute!"

**Major Tools**

- **Long Exact Sequence for relative homotopy groups**
  \[ (X, A, x_0) \]
  \[ \to \pi_n(A, x_0) \xrightarrow{id} \pi_n(X, x_0) \to \pi_n(X, A, x_0) \]
  \[ \to \pi_{n-1}(A, x_0) \xrightarrow{id} \pi_{n-1}(X, x_0) \to \pi_{n-1}(X, A, x_0) \]
2. Long exact sequence of a fibration

\[ F \rightarrow E \xrightarrow{p} B \]

Recall: A fibration is a map \( p: E \rightarrow B \) having the homotopy lifting property with respect to all spaces \( X \).

\[ \tilde{g}_c \leftarrow \bar{g} \downarrow \bar{g} \downarrow \tilde{g} \]

\[ g_c : X \rightarrow B \]

\[ \cdots \rightarrow \pi_n (F, x_0) \rightarrow \pi_n (E, x_0) \xrightarrow{p_*} \pi_n (B, b_0) \rightarrow \cdots \]

\[ \left( \pi_{n+1} (F, x_0) \rightarrow \pi_{n+1} (E, x_0) \rightarrow \pi_{n+1} (B, b_0) \rightarrow \cdots \right) \]

3. Excision for homotopy

Recall for homology

\[ X = A \cup B \]

\[ C = A \cap \partial B \]

\[ H_n (A, C) \cong H_n (X, B) \]

Not generally true for homotopy!

Example: \( S^2 \simeq S^3 \)

\[ \pi_4 (S^2, S^2) = \mathbb{Z} \oplus \mathbb{Z}_2 \]

\[ \pi_4 (S^3 \vee S^3) = \mathbb{Z}_2 \times \mathbb{Z}_2 \]
Propn Let \((A, C)\) be \(n\)-connected and \((\Theta, C)\) be \(m\)-connected.

Then \(T_k (A, C) \rightarrow T_{\Theta} (X, \Theta)\) is an isomorphism for \(k=n+m\), and a surjection for \(k=n+m\).

**Corollary (Freudenthal Suspension)** Let \(X\) be a CW complex.

Let \(SX = X \times [0,1] / (x,0) \sim (x_1,0)\). The map \(\tilde{\pi}_i (X) \rightarrow \tilde{\pi}_{i+1} (SX)\)

\((x, 1) \sim (x_1, 1)\)

is an isomorphism if \(X\) is \(n\)-connected for \(n\geq 2\) and \(i \geq 2n+1\), and a surjection if \(i = 2n+1\).

**Proof**

Let \(X\) be a CW complex with one zero-cell and otherwise cells of dimension \(\geq n\). Then the reduced cone complex \(C^+X\) is obtained from \(X\) by adding cells of dimension \(\geq n+1\). Similarly for \(C^-X\). Then \((C^+X, X)\) is \((n+1)\)-connected.

Now \(\tilde{\pi}_{i+1} (C^+X, X) \rightarrow \tilde{\pi}_{i+1} (EX, C^-X)\) is an isomorphism for \(i+1 < (n+1)+(n+1)\), i.e., \(i < 2n+1\), and a surjection for \(i = 2n+1\). So we have.
Corollary 5 $\pi_i^n (S^n) \cong \pi_{i+1}^{n+1} (S^{n+1})$ for $i < 2n - 1$.

In particular, $\pi_n (S^n) \cong \mathbb{Z}$.

Note that the proof of 5 has:
1. $\pi_n (S^n) \cong \pi_{n+1}^{n+1} (S^{n+1})$ for $n > 1$.
2. $\pi_1 (S^1) \cong \pi_2 (S^2)$ only known to be a surjection.

But we can easily write down a \( \mathbb{Z} \)'s worth of non-homotopic maps $S^2 \to S^2$, or more generally $S^n \to S^n$, using Brower degree.

Can use this to get to Hurewicz.

Proof The $i$th stable homotopy group of spheres (or stable $i$-stem) is the group $\pi_i^s (S^0) = \pi_{i+n} (S^n)$.

Note this implies the sequence of iterated suspensions $\pi_i (x) \to \pi_{i+1} (Sx) \to \pi_{i+2} (S^2x) \to \cdots$ eventually stabilizes in isomorphism.
Proof of Excision

Propn (Excision for homotopy)

\[ (A, C) \text{ n-cod} \]
\[ (B, C) \text{ m-cod} \]

\[ \pi_k(A, C) \to \pi_k(X, B) \text{ iso for } k \neq m, \]
\[ \text{surjection for } k = m. \]

Simple Case. Let \( C \) a CW cpk, \( A = C \cup e^m \), \( B = C \cup e^{m-1} \).

Let \( U, V \) open such that \( \bar{U} \subseteq \text{Int}(e^m) \). Given \( \bar{z} \in \pi_k(X, B) \), \( \bar{V} \subseteq \text{Int}(e^{m-1}) \)

make \( E \) smooth on \( \Sigma^{-1}(U) \) and \( \Sigma^{-1}(V) \).

\[ A = X - \varepsilon q \]
\[ B = X - \varepsilon p \]
\[ C = X - \varepsilon p, q \]

Thm \( C^\infty(x, y) \to C(x, y) \)
is a homotopy equivalence for smooth manifolds.

Sets are labelled by what they map to.
Pick \( p \in U, q \in V \) regular values. (If these don't exist, can homotop off at least one side)
\[ S = \mathbb{E}^{-1}(p) \text{ manifold of codim } n-1, \text{ dim } n-k-1 \]
\[ T = \mathbb{E}^{-1}(q) = \mathbb{S}^{m+1}, \text{ dim } k-m-1 \]

T can have boundary, S can't.

**Principle** There is no linking for manifolds of large codimension.

**Examples**

**Lemma** \[ S \sqcup T \leq \mathbb{R}^k \] smooth submanifolds w/ \( q+b < k-1 \). Then \( S \) and \( T \) are **split**, i.e. can be isotoped so \( S \) is to the left of \( x=0 \) and \( T \) is to the right.

**PF** Consider the projection \( p : \mathbb{R}^k \rightarrow \mathbb{R}^{k-1} \) forgetting first coordinate. \( p|_S \) and \( p|_T \) can be made transverse by a small perturbation so that \( p|_S(TS) \) and \( p|_T(TT) \) span \( \mathbb{R}^{k-1} \) at every point in their intersection. But this never actually happens, since \( a+b \leq k-1 \), so \( p(S) \cap p(T) = \emptyset \). Now pull \( S \) to the left and \( T \) to the right.
We apply this in our own case: Look at 
\((x, \emptyset) \cong (x, X \setminus \emptyset)\). Inside this space, \(C\) is homotopic to a 
map into \((X \setminus \emptyset, X \setminus \emptyset, \emptyset) \cong (A, C)\), via leftward sweep.

This shows \(\pi_k(A, C) \to \pi_k(X, A)\). To show injectivity, some argument

For a homotopy,

**More complicated step** For \(B - C\) a union of cells of dim \(n \geq m + 1\),
and \(A\) a single cell, \(C\) hits finitely many cells, so \(C \subseteq B\) for
sufficiently large \(b\). Can pull \(C\) down step-by-step. Can similarly
deal w/ the case that \(A = (U(n+1)\text{-cells}) \cup C\); the image
of \(C\) hits finitely many, and we can do the unlinking above
one at a time.

Let \(A_i = (i\text{-skeleton of } A) \cup C\)
\(X_i = (i\text{-skeleton of } A) \cup B\)

\[\text{wts} \quad \pi_k(A_{i+1}, C) \to \pi_k(x_{i+1}, B)\] has the desired properties.
\[ \pi_k(A_i, A_j) \rightarrow \pi_k(A_i, c) \rightarrow \pi_k(A_i, A_j) \rightarrow \pi_{k-1}(A_i, c) \rightarrow \cdots \]

\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]

\[ \pi_{k+1}(x_{i+1}, x_{i}^c) \rightarrow \pi_k(x_{i+1}, c) \rightarrow \pi_k(x_{i+1}, c) \rightarrow \pi_{k-1}(x_i, c) \rightarrow \cdots \]

By base case

\((A_i, A_j)\) is \(i\)-ctdl,

in particular \((n+1)\)-ctdl

\((x_i, A_i)\) is \(m\)-ctdl

*Repeat, ascending in dimension.

Maps are iso when \(k = m\)

* When \(k = m\), second and first maps are surjective, fourth and fifth maps are iso, showing

\[ \pi_k(A_i, i) \rightarrow \pi_k(x_{i+1}, c) \]