- All spaces are Hausdorff
- All maps are cts

Math 961: Lecture 1

Last semester you introduced the homotopy groups

$$\pi_n(X, x_0) = \{f : (S^n, s) \longrightarrow (X, x_0)\} / \sim \text{ homotopy}$$

Why care?

- Natural, easily defined
- Powerful

eg Whitehead's Theorem X, Y ctd CW complexes and $\exists f : X \to Y$ st $f_* : \pi_n(X, x_0) \xrightarrow{\sim} \pi_n(Y, y_0)$ for all n, then f is a homotopy equivalence.

Hurewicz X w/ $\tilde{H}_i(X) = 0$ for $i < n$, where $n > 1$. Then $\pi_i(X) = 0$ for $i < n$, and $\pi_n(X) = \tilde{H}_n(X)$.

- But hard to compute!

Major Tools

① Long Exact sequence for relative homotopy groups

(X, A, x_0)

$$\hookrightarrow \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \longrightarrow \pi_n(X, A, x_0) \rightarrow$$

$$\hookrightarrow \pi_{n-1}(A, x_0) \xrightarrow{i_*} \pi_{n-1}(X, x_0) \longrightarrow \pi_{n-1}(X, A, x_0) \rightarrow$$

2 Long exact sequence of a fibration

$$\begin{array}{c} F \hookrightarrow E \\ \downarrow p \\ B \end{array}$$

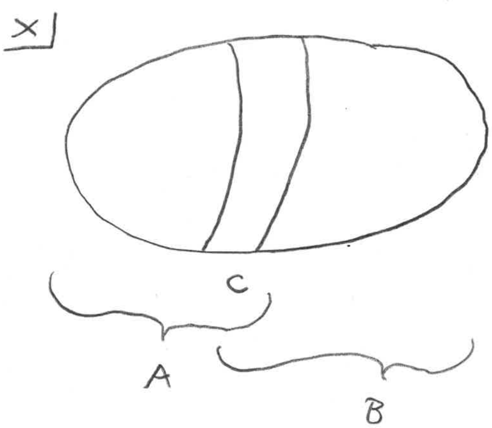
Recall A Fibration is a map $p: E \to B$ having the homotopy lifting property with respect to all spaces X.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{g}_0"] \arrow[ur, dashed, bend left, "\tilde{g}_t"] \arrow[r, "g_t"'] & B \end{tikzcd}$$

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \pi_{n-1}(E, x_0) \to \pi_{n-1}(B, b_0) \to \cdots$$

3 Excision for homotopy

Recall for homology



$X = A \cup B$

$C = A \cap B$

$H_n(A, C) \cong H_n(X, B)$

Not generally true for homotopy!

Example $S^2 \subseteq S^3$ $\quad \pi_4(S^3, S^2) = \mathbb{Z} \oplus \mathbb{Z}_2$

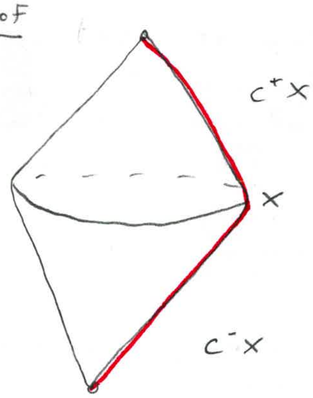
$\pi_4(S^3 \vee S^3) = \mathbb{Z}_2 \times \mathbb{Z}_2$

<u>Propn</u> Let (A, C) be n-connected and (B, C) be m-conneted

Then $\pi_k(A, C) \longrightarrow \pi_k(X, B)$ is an isomorphism for $k < n+m$, and a surjection for $k = n+m$.

$\rightsquigarrow$ <u>Corollary</u> (Freudenthal Suspension) Let X be a CW cpx. Let $SX = X \times [0,1] \Big/ \begin{smallmatrix}(x_1,0)\sim(x_2,0)\\ (x_1,1)\sim(x_2,1)\end{smallmatrix}$. The map $\pi_i(X) \to \pi_{i+1}(SX)$ is an isomorphism if X is n-connected for $n \geq 2$ and $i < 2n+1$, and a surjection if $i = 2n+1$.

<u>Proof</u>



Best to work w/ the reduced suspension ΣX and reduced cone complex C^+X. These are homotopy equivalent for CW cpxes.

Let X be a CW cpx w/ one zero-cell and otherwise cells of dimn $> n$. Then the reduced cone complex C^+X is obtained from X by adding cells of dimn $> n+1$. Similarly for C^-X. Then $(C^{\pm}X, X)$ is $(n+1)$-connected.

Now $\pi_{i+1}(C^+X, X) \longrightarrow \pi_{i+1}(\Sigma X, C^-X)$ is an isomorphism for $i+1 < (n+1)+(n+1)$, i.e. $i < 2n+1$, and a surjection for $i = 2n+1$. So we have

$$
\begin{tikzcd}
\pi_i(X) \arrow[r, "\text{les of the pair}", "\simeq"'] \arrow[ddr, bend right] & \pi_{i+1}(C^+X, X) \arrow[d, "\leftarrow \text{excision for homotopy}"] \\
& \pi_{i+1}(\Sigma X, C^-X) \arrow[d, "\leftarrow C^-X \text{ is contractible}"] \\
& \pi_{i+1}(\Sigma X)
\end{tikzcd}
$$

$\square$

<u>Corollary</u> (a) $\pi_i(S^n) \simeq \pi_{i+1}(S^{n+1})$ for $i < 2n-1$.

(b) In particular $\pi_n(S^n) = \mathbb{Z}$

Note that the proof of (b) has:

- $\pi_n(S^n) \xrightarrow{\sim} \pi_{n+1}(S^{n+1})$ is for $n > 1$
- $\pi_1(S^1) \twoheadrightarrow \pi_2(S^2)$ only known to be a surjection.

But we can easily write down a $\mathbb{Z}$'s worth of non-homotopic maps $S^2 \to S^2$, or more generally $S^n \to S^n$, using Brower degree.

- Can use this to get to Hurewicz.

<u>Defn</u> The ith stable homotopy group of spheres (or stable i-stem) is the group $\pi_i^s(S^0) = \pi_{i+n}(S^n)$

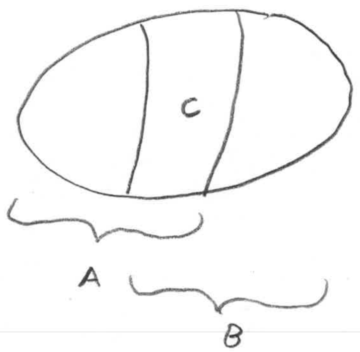
Note this implies the sequence of iterated suspensions

$\pi_i(X) \to \pi_{i+1}(SX) \to \pi_{i+2}(S^2X) \to \cdots$ eventually stabilizes in isomorphisms

$\pi_i^s(X)$ is the ith stable homotopy group

Proof of Excision

Propn (Excision for homotopy)



(A, C) n-ctd

(B, C) m-ctd

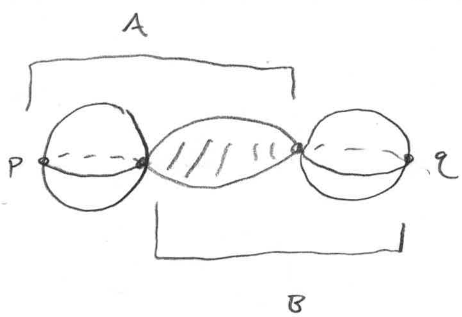
$\pi_k(A,C) \longrightarrow \pi_k(X, B)$ iso for $k < n+m$,

surjection for $k = n+m$

Simple Case Let C a CW cpx, $A = C \cup e^{n+1}$, $B = C \cup e^{m+1}$.

Let U, V open such that $\bar{U} \subseteq \mathrm{Int}(e^{n+1})$, $\bar{V} \subseteq \mathrm{Int}(e^{m+1})$. Given $[e] \in \pi_k(X, B)$,

make e smooth on $e^{-1}(U)$ and $e^{-1}(V)$.

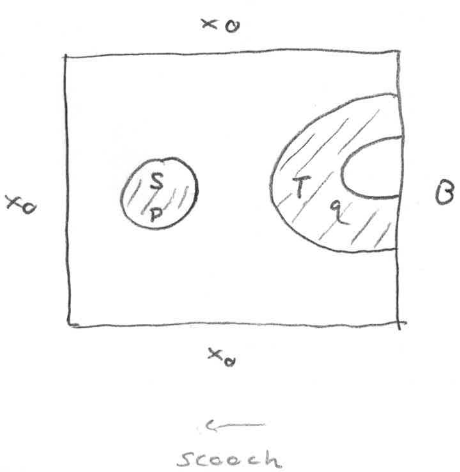


$A \simeq X - \{q\}$

$B \simeq X - \{p\}$

$C \simeq X - \{p, q\}$

> **Thm** $e^{\infty}(X, Y) \hookrightarrow e(X, Y)$ is a homotopy equivalence for smooth manifolds.



Sets are labelled by what they map to

Pick $p \in U$, $q \in V$ regular values. (If these don't exist, can homotop off at least one side

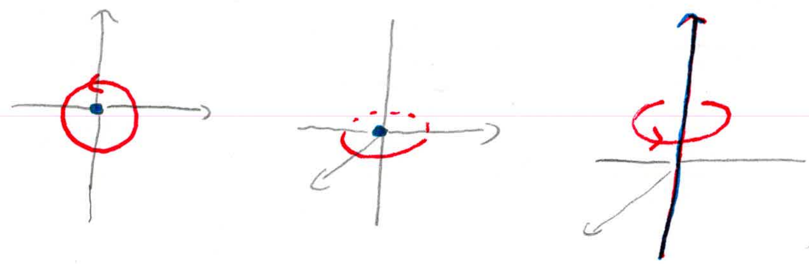
$S = \varphi^{-1}(p)$ manifold of codimn $n+1$, dimn $k-n-1$

$T = \varphi^{-1}(q)$ " " " $m+1$, dimn $k-m-1$

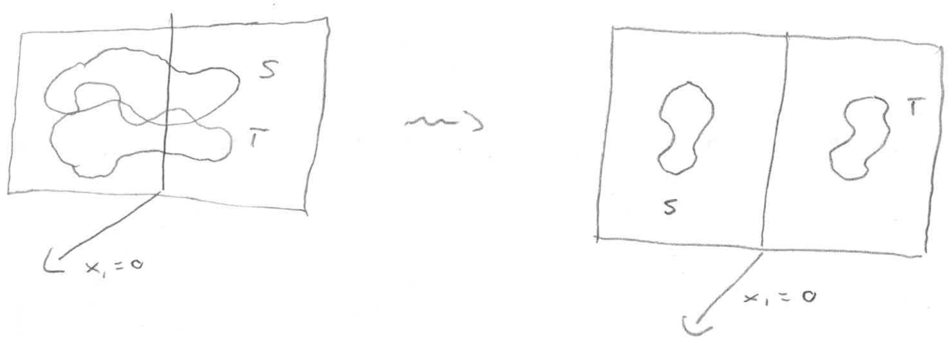
T can have boundary, S can't.

<u>Principle</u> There is no linking for manifolds of large codimension.

<u>Examples</u>



<u>Lemma</u> $S^a, T^b \subseteq \mathbb{R}^k$ smooth submanifolds w/ $a+b < k-1$. Then S and T are <u>split</u>, i.e. can be isotoped so S is to the left of $x_1 = 0$ and T is to the right.



<u>PF</u> Consider the projection $p : \mathbb{R}^k \to \mathbb{R}^{k-1}$ forgetting first coordinate. $p|_S$ and $p|_T$ can be made transverse by a small perturbation so that $p_*(TS)$ and $p_*(TT)$ span $\mathbb{R}^{k-1}$ at every point in their intersection. But this never actually happens, since $a+b < k-1$. So $p(S) \cap p(T) = \phi$. Now pull S to the left and T to the right.

We apply this in our own case: Look at

$(X, B) \simeq (X, X \setminus \{p\})$. Inside this space, φ is homotopic to a map into $(X \setminus \{q\}, X \setminus \{p, q\}) \simeq (A, C)$, via leftward scooch.

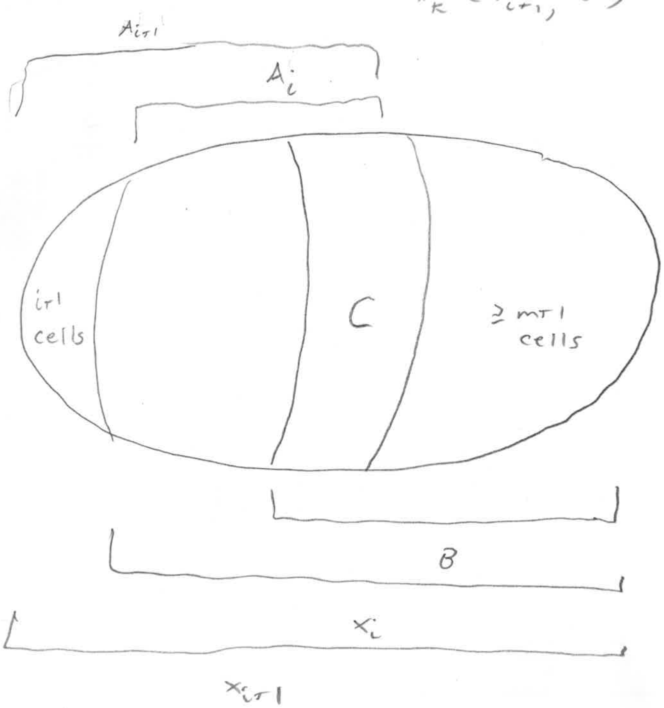
This shows $\pi_k(A, C) \to \pi_k(X, B)$. To show injectivity, same argument for a homotopy.

<u>More complicated step</u> For $B - C$ a union of cells of dimn $\geq m+1$, and A a single cell, φ hits finitely many cells, so $\varphi \subseteq B^{\ell}$ for sufficiently large ℓ. Can pull φ down step-by-step. Can similarly deal w/ the case that $A = (\cup (n+1)\text{-cells}) \cup C$; the image of φ hits finitely many, and we can do the unlinking above one at a time.

Let $A_i = (i\text{-skeleton of } A) \cup C$

$X_i = (i\text{-skeleton of } A) \cup B$

<u>Wts</u> $\pi_k(A_{i+1}, C) \to \pi_k(X_{i+1}, B)$ has the desired properties.



$k < m+n$)

$$\begin{tikzcd}
\to \pi_{k+1}(A^i_{i+1}, A^i_i) \arrow[r] \arrow[d, "S"'] & \pi_k(A^i_i, C) \arrow[r] \arrow[d, "S"'] & \pi_k(A^i_{i+1}, C) \arrow[r] \arrow[d] & \pi_k(A_{i+1}, A_i) \arrow[r] \arrow[d] & \pi_{k-1}(A_i, C) \arrow[r] \arrow[d] & \cdots \\
\to \pi_{k+1}(X_{i+1}, X_i) \arrow[r] & \pi_k(X_i, B) \arrow[r] & \pi_k(X_{i+1}, B) \arrow[r] & \pi_k(X_{i+1}, X_i) \arrow[r] & \pi_{k-1}(X_i, B) \arrow[r] & \cdots
\end{tikzcd}$$

$\uparrow$

By base case

(A^i_{i+1}, A^i_i) is i-ctd,

in particular $(n+1)$-ctd

(X_i, A_i) is m-ctd

$\uparrow$

Base case

stays iso

at k

$\uparrow$

Induction

- Repeat, ascending in dimension.
- Maps are iso when $k < m+n$
- When $k = m+n$, second and first maps are surjective, fourth and fifth maps are iso, showing $\pi_k(A^{i+1}_{i+1}) \twoheadrightarrow \pi_k(X^i_{i+1}, B)$.