APPENDIX D

Equivalence Relations

This appendix may be read anytime after you've finished Appendix B, but it is not needed in the text until Section 9.4. If you read it before that point, you should have no trouble with Examples 1–3 but may have to skip some of the unnumbered examples. Chapter 2 is a prerequisite for the examples labeled "integers," Chapter 6 for those labeled "rings," and Chapter 7 for those labeled "groups."

If A is a set, then any subset of $A \times A$ is called a **relation** on A. A relation T on A is called an **equivalence relation** provided that the subset T is

(i) **Reflexive:** $(a,a) \in T$ for every $a \in A$.
(ii) **Symmetric:** If $(a,b) \in T$, then $(b,a) \in T$.
(iii) **Transitive:** If $(a,b) \in T$ and $(b,c) \in T$, then $(a,c) \in T$.

If T is an equivalence relation on A and $(a,b) \in T$, we say that ***a* is equivalent to *b*** and write $\boldsymbol{a \sim b}$ instead of $(a,b) \in T$. In this notation, the conditions defining an equivalence relation become

(i) **Reflexive:** $a \sim a$ for every $a \in A$.
(ii) **Symmetric:** If $a \sim b$, then $b \sim a$.
(iii) **Transitive:** If $a \sim b$ and $b \sim c$, then $a \sim c$.

When this notation is used, the relation is usually defined without explicit reference to a subset of $A \times A$.

EXAMPLE 1 Let A be a set and define $a \sim b$ to mean $a = b$. In other words, the equivalence relation on A is the subset $T = \{(a,b) \mid a = b\}$ of $A \times A$. Then it is easy to see that $=$ is an equivalence relation.

EXAMPLE 2 The relation on the set $\mathbb{R}$ of real numbers defined by

$$r \sim s \text{ means } |r| = |s|$$

is an equivalence relation, as you can readily verify.

EXAMPLE 3* Define a relation on the set $\mathbb{Z}$ of integers by

$$a \sim b \text{ means } a - b \text{ is a multiple of 3.}$$

For example, $17 \sim 5$ since $17 - 5 = 12$, a multiple of 3. Clearly $a \sim a$ for every a since $a - a = 0 = 3 \cdot 0$. To prove property (ii), suppose $a \sim b$. Then $a - b$ is a multiple of 3. Hence, $-(a - b)$ is also a multiple of 3. But $-(a - b) = b - a$. Therefore, $b \sim a$. To prove property (iii), suppose $a \sim b$ and $b \sim c$. Then $a - b$ and $b - c$ are multiples of 3 and so is their difference $(a - b) - (b - c) = a - c$, so that $a \sim c$. Thus $\sim$ is an equivalence relation (usually called congruence modulo 3 and denoted $a \equiv b \pmod 3$).

EXAMPLE (INTEGERS) If n is a fixed positive integer, the relation of congruence modulo n on the set $\mathbb{Z}$, defined by

$$a \equiv b \pmod n \text{ if and only if } a - b \text{ is a multiple of } n,$$

is an equivalence relation by Theorem 2.1.

EXAMPLE (RINGS) If I is an ideal in the ring R, then the relation of congruence modulo I, defined by

$$a \equiv b \pmod I \text{ if and only if } a - b \in I,$$

is an equivalence relation on R by Theorem 6.4.

EXAMPLE (GROUPS) If K is a subgroup of a group G, then the relation defined by

$$a \equiv b \text{ if and only if } ab^{-1} \in K$$

is an equivalence relation on G by Theorem 7.22.

Warning It is quite possible to have a relation on a set that satisfies one or two, but not all three, of the properties that define an equivalence relation. For instance, the order relation $\leq$ on the set $\mathbb{R}$ of real numbers is reflexive and transitive but not symmetric; for other examples, see Exercises 8 and 9. Therefore, you must verify all three properties in order to prove that a particular relation is actually an equivalence relation.

* If you've already read Section 2.1, skip Example 3 here and below; it's just congruence modulo n when $n = 3$.

Let $\sim$ be an equivalence relation on a set A. If $a \in A$, then the **equivalence class** of a (denoted $[a]$) is the set of all elements in A that are equivalent to a, that is,

$$[a] = \{b \mid b \in A \text{ and } b \sim a\}.$$

In Example 2, for instance, the equivalence class $[9]$ of the number 9 consists of all real numbers b such that $b \sim 9$, that is, all numbers b such that $|b| = |9|$. Thus $[9] = \{9, -9\}$.

EXAMPLE (RINGS, GROUPS) If I is an ideal in a ring R, then an equivalence class under the relation of congruence modulo I is a coset $a + I = \{a + i \mid i \in I\}$. Similarly, if K is a subgroup of a group G, then an equivalence class of the relation congruence modulo K is a right coset $Ka = \{ka \mid k \in K\}$.

EXAMPLE 3 (continued) The equivalence class of the integer 2 consists of all integers b such that $b \sim 2$, that is, all b such that $b - 2$ is a multiple of 3. But $b - 2$ is a multiple of 3 exactly when b is of the form $b = 2 + 3k$ for some integer k. Therefore,

$$[2] = \{2 + 3k \mid k \in \mathbb{Z}\} = \{2 + 0, 2 \pm 3, 2 \pm 6, 2 \pm 9, \ldots\}$$
$$= \{\ldots, -7, -4, -1, 2, 5, 8, 11, \ldots\}.$$

A similar argument shows that the equivalence class $[8]$ consists of all integers of the form $8 + 3k$ $(k \in \mathbb{Z})$; consequently,

$$[8] = \{\ldots, -7, -4, -1, 2, 5, 8, 11, 14, 17, \ldots\}.$$

Thus $[2]$ and $[8]$ are the same set. Note that $2 \sim 8$. This is an example of

THEOREM D.1 *Let $\sim$ be an equivalence relation on a set A and $a, b \in A$. Then*

$$a \sim c \textit{ if and only if } [a] = [c].$$

Proof* Assume $a \sim c$. To prove that $[a] = [c]$, we first show that $[a] \subseteq [c]$. To do this, let $b \in [a]$. Then $b \sim a$ by definition. Since $a \sim c$, we have $b \sim c$ by transitivity. Therefore, $b \in [c]$ and $[a] \subseteq [c]$. Reversing the roles of a and c in this argument and using the fact that $c \sim a$ by symmetry, show that $[c] \subseteq [a]$. Therefore, $[a] = [c]$. Conversely, assume that $[a] = [c]$. Since $a \sim a$ by reflexivity, we have $a \in [a]$, and, hence, $a \in [c]$. The definition of $[c]$ shows that $a \sim c$. ◆

* If you've read Section 2.1, note that this proof and the proof of Corollary D.2 are virtually identical to the proofs of Theorem 2.3 and Corollary 2.4: Just replace $\equiv$ by $\sim$.

Generally when one has two sets, there are three possibilities: The sets are equal, the sets are disjoint, or the sets have some (but not all) elements in common. With equivalence classes, the third possibility cannot occur:

COROLLARY D.2 *Let $\sim$ be an equivalence relation on a set A. Then any two equivalence classes are either disjoint or identical.*

Proof Let $[a]$ and $[c]$ be equivalence classes. If they are disjoint, then there is nothing to prove. If they are not disjoint, then $[a] \cap [c]$ is nonempty, and by definition there is an element b such that $b \in [a]$ and $b \in [c]$. By the definition of equivalence class, $b \sim a$ and $b \sim c$. Consequently, by transitivity and symmetry, $a \sim c$. Therefore, $[a] = [c]$ by Theorem D.1. ◆

A **partition** of a set A is a collection of nonempty, mutually disjoint* subsets of A whose union is A. Every equivalence relation $\sim$ on A leads to a partition as follows. Since $a \in [a]$ for each $a \in A$, every equivalence class is nonempty, and every element of A is in one. Distinct equivalence classes are disjoint by Corollary D.2. Therefore,

The distinct equivalence classes of an equivalence relation on a set A form a partition of A.

Conversely, every partition of A leads to an equivalence relation whose equivalence classes are precisely the subsets of the partition (Exercise 21).

◆ EXERCISES

A.

1. Let P be a plane. If p, q are points in P, then $p \sim q$ means p and q are the same distance from the origin. Prove that $\sim$ is an equivalence relation on P.

2. Define a relation on the set $\mathbb{Q}$ of rational numbers by: $r \sim s$ if and only if $r - s \in \mathbb{Z}$. Prove that $\sim$ is an equivalence relation.

3. **(a)** Prove that the following relation on the set $\mathbb{R}$ of real numbers is an equivalence relation: $a \sim b$ if and only if $\cos a = \cos b$.

(b) Describe the equivalence class of 0 and the equivalence class of $\pi/2$.

4. If m and n are lines in a plane P, define $m \sim n$ to mean that m and n are parallel. Is $\sim$ an equivalence relation on P?

5. **(a)** Let $\sim$ be the relation on the ordinary coordinate plane defined by $(x,y) \sim (u,v)$ if and only if $x = u$. Prove that $\sim$ is an equivalence relation.

(b) Describe the equivalence classes of this relation.

* That is, any two of the subsets are disjoint.

B.

6. Prove that the following relation on the coordinate plane is an equivalence relation: $(x,y) \sim (u,v)$ if and only if $x - u$ is an integer.

7. Let $f: A \to B$ be a function. Prove that the following relation is an equivalence relation of $A: u \sim v$ if and only if $f(u) = f(v)$.

8. Let $A = \{1, 2, 3\}$. Use the ordered-pair definition of a relation to exhibit a relation on A with the stated properties.

   (a) Reflexive, not symmetric, not transitive.

   (b) Symmetric, not reflexive, not transitive.

   (c) Transitive, not reflexive, not symmetric.

   (d) Reflexive and symmetric, not transitive.

   (e) Reflexive and transitive, not symmetric.

   (f) Symmetric and transitive, not reflexive.

9. Which of the properties (reflexive, symmetric, transitive) does the given relation have?

   (a) $a < b$ on the set $\mathbb{R}$ of real numbers.

   (b) $A \subseteq B$ on the set of all subsets of a set S.

   (c) $a \neq b$ on the set $\mathbb{R}$ of real numbers.

   (d) $(-1)^a = (-1)^b$ on the set $\mathbb{Z}$ of integers.

B. 10. If r is a real number, then $[\![r]\!]$ denotes the largest integer that is $\leq r$; for instance $[\![\pi]\!] = 3$, $[\![7]\!] = 7$ and $[\![-1.5]\!] = -2$. Prove that the following relation is an equivalence relation on $\mathbb{R}$: $r \sim s$ if and only if $[\![r]\!] = [\![s]\!]$.

11. Let $\sim$ be defined on the set $\mathbb{R}^*$ of nonzero real numbers by: $a \sim b$ if and only if $a/b \in \mathbb{Q}$. Prove that $\sim$ is an equivalence relation.

12. Is the following relation an equivalence relation on $\mathbb{R}$: $a \sim b$ if and only if there exists $k \in \mathbb{Z}$ such that $a = 10^k b$.

13. In the set $\mathbb{R}[x]$ of all polynomials with real coefficients, define $f(x) \sim g(x)$ if and only if $f'(x) = g'(x)$, where $'$ denotes the derivative. Prove that $\sim$ is an equivalence relation on $\mathbb{R}[x]$.

14. Let T be the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and define $f \sim g$ if and only if $f(2) = g(2)$. Prove that $\sim$ is an equivalence relation.

15. Prove that the relation on $\mathbb{Z}$ defined by $a \sim b$ if and only if $a^2 \equiv b^2 \pmod 6$ is an equivalence relation.

16. Let $S = \{(a,b) \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\}$ and define $(a,b) \sim (c,d)$ if and only if $ad = bc$. Prove that $\sim$ is an equivalence relation on S.

17. Let $\sim$ be a symmetric and transitive relation on a set A. What is wrong with the following "proof" that $\sim$ is reflexive: $a \sim b$ implies $b \sim a$ by symmetry; then $a \sim b$ and $b \sim a$ imply $a \sim a$ by transitivity. [Also see Exercise 8(f).]

18. Let G be a group and define $a \sim b$ if and only if there exists $c \in G$ such that $b = c^{-1}ac$. Prove that $\sim$ is an equivalence relation on G.

19. **(a)** Let K be a subgroup of a group G and define $a \sim b$ if and only if $a^{-1}b \in K$. Prove that $\sim$ is an equivalence relation on G.

    (b) Give an example to show that the equivalence relation in part (a) need not be the same as the relation in the last example on page 528.

20. Let G be a subgroup of S_n. Define a relation on the set $\{1, 2, \ldots, n\}$ by $a \sim b$ if and only if $a = \sigma(b)$ for some σ in G. Prove that $\sim$ is an equivalence relation.

21. Let A be a set and $\{A_i \mid i \in I\}$ a partition of A. Define a relation on A by: $a \sim b$ if and only if a and b are in the same subset of the partition (that is, there exists $k \in I$ such that $a \in A_k$ and $b \in A_k$).

    (a) Prove that $\sim$ is an equivalence relation on A.

    (b) Prove that the equivalence classes of $\sim$ are precisely the subsets A_i of the partition.