Homework 8 Solutions

MTH 327H

4.
- Suppose that $[0,1] = \bigcup_{i=1}^{\infty} A_i$, and each A_i is closed. Then consider $A_i^c = U_i$, which is open. If every A_i has empty interior, we see that U_i is dense, so by the Baire Category Theorem, $\bigcap_{i=1}^{\infty} U_i$ is dense in $\mathbb{R}$. But this is nonsense, since $[0,1] \cap (\bigcap_{i=1}^{\infty} U_i)$ is empty. Ergo some A_i has nonempty interior.
- Observe that $\mathbb{R} - \mathbb{Q}$ may be written as the intersection of the countably many open dense subsets $U_q = \mathbb{R} - \{q\}$ for $q \in \mathbb{Q}$. Suppose that $\mathbb{Q}$ may be written as the intersection of countably many open sets V_i, each of which is necessarily dense in $\mathbb{R}$ since it contains $\mathbb{Q}$. Then the intersection of the countably many open dense sets in the collection $\{U_q : q \in \mathbb{Q}\} \cup \{V_i : i \in \mathbb{N} - \{0\}\}$ is empty. This is impossible by the Baire Category Theorem. Hence $\mathbb{Q}$ cannot be written as the intersection of countably many open sets.

5.
- We see that $s(0)$ converges trivially. For any other $x \in [-1,1]$, $s(x)$ is an alternating series. The terms $\frac{|x|^{2n+1}}{2n+1}$ decrease monotonically to 0, so $s(x)$ converges by the Alternating Series Test.

  If we assume we can differentiate $s(x)$ term by term on $(-1,1)$ (which is a big assumption!) we see that

$$\begin{aligned} s'(x) &= 1 - x^2 + x^4 - x^6 + \dots \\ &= \sum_{n=0}^{\infty} (-x^2)^n \\ &= \frac{1}{1+x^2} \end{aligned}$$

  This suggests that $s(x) = \arctan x$. If we assume this expansion is valid on $[-1,1]$ (again, big assumption), we get that

$$\frac{\pi}{4} = \arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

- Let $b_i = |a_i|$ and recall that (b_i) is a monotonically decreasing sequence with $b_i \to 0$. Then let s_i be the partial sums of (a_i), so that we have

$$s_2 < s_4 < s_6 \dots < \frac{\pi}{4} < \dots s_5 < s_3 < s_1$$

  Let $t_i = \frac{s_i + s_{i+1}}{2} = s_i + \frac{a_{i+1}}{2}$. We observe that the limit of the sequence (t_i) is $\lim(s_i + \frac{a_{i+1}}{2}) = \frac{\pi}{4} + 0$. Now we look at the even-index terms in this sequence. For any i, we see that

$$t_{2i+2} = s_{2i+2} + \frac{a_{2i+3}}{2} = s_{2i+2} + \frac{b_{2i+3}}{2} \geqslant s_{2i} + \frac{b_{2i+1}}{2}.$$

So the even partial sums form an increasing sequence $t_2 < t_4 < t_6 < \dots$; since the limit of this sequence is $\frac{\pi}{4}$, each even partial sum must in fact be less than $\frac{\pi}{4}$, so we have $t_2 < t_4 < t_6 < \cdots < \frac{\pi}{4}$. Similarly we see that the odd partial sums are a decreasing sequence, giving us

$$t_2 < t_4 < t_6 \cdots < \frac{\pi}{4} < \dots t_5 < t_3 < t_1$$

The only fact we used about the (s_i) to reach this conclusion is that $s_i - s_{i+1}$ is an alternating sequences which decreases montonically in absolute value to zero; as this is plainly still true of the t_i, it follows that we may iterate this argument.