MTH 327H: Homework 8

Due: November 2, 2018

1. Office hours the tenth week of classes are W 11:30-12:30 and 3-4, and Th 9-10. I will be at Princeton on Monday and Tuesday; Monday's lecture will be given by Professor Angelini-Knoll.
2. Read Rudin Sections 4.1-21.
3. Do problems 9, 13, 14, 16, 23 in Rudin Chapter 3.
4. This question is a follow-up to one of the problems from last week.
   - Show that if $[0,1]$ is expressed as a union of countably many closed sets $\{A_i\}$ then at least one A_i has nonempty interior.
   - Show that $\mathbb{Q}$ cannot be written as the intersection of countably many open subsets of $\mathbb{R}$.
5. Let
$$s(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$
   - Prove this power series converges on $(-1,1] \in \mathbb{R}$. Give a *non-rigorous* argument that
$$s'(x) = \frac{1}{1+x^2}$$
   and therefore that
$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \arctan(1) = \frac{\pi}{4}$$
   (Reasons this is non-rigorous currently include (1) we don't know what a derivative is, (2) we don't know how to differentiate power series, (3) we don't know what π is.)
   - Let
$$a_1 = 1 \qquad a_2 = -1/3 \qquad a_3 = 1/5 \qquad a_4 = -1/7 \qquad \dots$$
   and consider the partial sums
$$s_1 = a_1 \qquad s_2 = a_1 + a_2 \qquad s_3 = a_1 + a_2 + a_3 \qquad \dots$$
   Recall from class that
$$s_2 < s_4 < s_6 \cdots < \frac{\pi}{4} < \dots s_5 < s_3 < s_1.$$
   Show that if $t_i = \frac{s_i + s_{i+1}}{2}$, then we have
$$t_2 < t_4 < t_6 \cdots < \frac{\pi}{4} < \dots t_5 < t_3 < t_1,$$
   and if $u_i = \frac{t_i + t_{i+1}}{2}$, then
$$u_2 < u_4 < u_6 \cdots < \frac{\pi}{4} < \dots u_5 < u_3 < u_1$$
   and so on for further averages of the previous estimates.
   - Use the previous part of this problem and the first ten terms of the sequence $\{a_i\}$ to give an estimate for π.