Homework 7 Solutions

MTH 327H

4. Let $\{s_n\}$ be a sequence of real numbers, and let $S \subset \mathbb{R} \cup \{\pm\infty\}$ be the set of subsequential limits of $\{s_n\}$. Let $E_N = \{s_n : n \geqslant N\}$.

   We check the statement for the two definitions of lim sup; lim inf is similar. Suppose that $t = \lim_{N\to\infty} \sup E_N$. Then we claim t is exactly $\sup S$.

   First we do the case where t is a real number. Recall that to show t is a subsequential limit of $\{s_n\}$, it suffices to show that any neighborhood $(t-\epsilon, t+\epsilon)$ of t contains s_n for infinitely many n. Now, for any $\epsilon > 0$ we see that there exists N such that $N_0 \geqslant N$ implies that $\sup\{s_n : N_0 \geqslant N\} < t + \epsilon$, implying in particular that $\{\sup s_n : N_0 \geqslant N_0\} < t + \epsilon$. Indeed we see that for $n \geqslant N_0$, $s_n < t + \epsilon$. Furthermore, suppose that there are only finitely many $m_1, \dots, m_k$ for which $s_{m_i} > t - \epsilon$. Then for $n > N_1 = \max\{m_1, \dots, m_k\}$, we have that $s_n < t + \epsilon$, implying that for $N_0 > N_1$, $\sup\{s_n : n \geqslant N_0\} \leqslant t - \epsilon$, contradicting $\sup E_N \to t$. So there must be infinitely many n for which s_n is in $(t-\epsilon, t+\epsilon)$. We conclude that $t \in S$.

   We now wish to show that $t = \max S$. But let r be another subsequential limit, say of the subsequence $\{s_{n_k}\}$. Then $r = \lim s_{n_k} = \limsup s_{n_k} = \limsup\{s_{n_k} : n_k \geqslant N\} \leqslant \limsup\{s_n : n \geqslant N\} = t$. So t is $\max S$.

   Now let $t = -\infty$. This implies that for every $m < 0$, there is an N such that $N_0 \geqslant N$ implies that $\sup\{s_n : n \geqslant N_0\} < m$. In particular for $n \geqslant N$, $s_n < M$. So $s_n \to -\infty$ has exactly one subsequential limit, namely t.

   Now let $t = \infty$. Certainly if $t \in S$ then $t = \sup S$. Now pick N_1 such that for $N_0 \geqslant N_1$, $\sup\{s_n : n \geqslant N_0\} > 1$. In particular $\sup\{s_n : n \geqslant N_1\} > 1$, implying that there exists an $s_{n_1} > 1$ in the set. Now pick $N_2 > n_1$ such that $\sup\{s_n : n \geqslant N_2\} > 2$, and pick s_{n_2} in this set. Iterate to get a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ converging to ∞.

5. Let $s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{2^n}$. The sequence (s_n) is increasing bounded above; it's not hard to satisfy myself inductively that $s_n < 2$. So $\{s_n\}$ converges. Therefore, by Theorem 3.3, the sequence $\{2s_n - s_n\}$ converges to $2s - s = s$. But $2s_n - s_n = 2$ for any n. So $s = 2$.

   In the case of $t_n = 1 + 2 + \cdots 2^n$, Theorem 3.3 does not apply.