Homework 4 Solutions

MTH 327H

4. Suppose that $\boxplus$ is an order on $\mathbb{Q}$ that gives $\mathbb{Q}$ the structure of an ordered field. By Proposition 1.18(d), $0 \boxplus 1$. Adding 1 to both sides preserves the ordering by Definition 1.17(i), so $1 \boxplus 2$. Iterating this process, we see that $n-1 \boxplus n$. Since $\boxplus$ is transitive we have now ordered the naturals:

$$0 \boxplus 1 \boxplus 2 \boxplus 3 \ldots$$

By Proposition 1.18(a), taking additive inverses reverses ordering, so we know the entire ordering of the integers:

$$\cdots - 2 \boxplus -1 \boxplus 0 \boxplus 1 \boxplus 2 \boxplus 3 \ldots$$

In particular, we conclude that if x and y are integers, $x \boxplus y$ if and only if $x < y$.

Now suppose that we have two rationals $r < s$. We may write $r = \frac{a}{b}$, $s = \frac{c}{b}$ with $b > 0$ after possibly taking common denominators if necessary. Then $r < s \Leftrightarrow a < c \Leftrightarrow a \boxplus c \Leftrightarrow \frac{a}{b} \boxplus \frac{c}{b}$, where the last step follows by observing that $0 < b \Leftrightarrow 0 \boxplus b \Leftrightarrow 0 \boxplus \frac{1}{b}$ by Proposition 1.18(e) and multiplication by positives preserves order by Proposition 1.18(b). We conclude that $\boxplus$ is the same order as $<$.

5. Suppose that $\mathbb{C}$ can be given the structure of an ordered field. Then since squares are positive in any ordered field, $0 < 1$ and $0 < -1 = i^2$. But taking additive inverses reverse order in any ordered field, so this is impossible.

Now suppose that $\mathbb{Z}/p\mathbb{Z}$ can be given the structure of an ordered field. Then $[0] < [1]$, since this is true in any ordered field. But then adding $[1]$ to both sides shows that $[1] < [2]$, and iterating this process and applying transitivity we see that $[0] < [1] < \cdots < [p-2] < [p-1]$. In particular $[0] < [p-1]$. But adding $[1]$ to both sides of $[p-2] < [p-1]$ shows that $[p-1] < [p] = [0]$. This is a contradiction. So $\mathbb{Z}/p\mathbb{Z}$ cannot be given the structure of an ordered field.

6. (a) Suppose that $a_1, \cdots a_{n-1}$ have been chosen as specified, and that a_n is the largest integer such that

$$a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + \cdots + \frac{a_n}{k^n} \leqslant x$$

First suppose that $a_n < 0$. This implies that

$$a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + \cdots + \frac{0}{k^n} > x$$

which is impossible by construction of a_{n-1}. So $a_n \geqslant 0$. Now suppose $a_n \geqslant k$. Then $\frac{a_n}{k^n} > \frac{k}{k^n} = \frac{1}{k^{n-1}}$. This implies that

$$a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + \cdots + \frac{a_{n-1}}{k^{n-1}} + \frac{1}{k^{n-1}} \leqslant x$$

so in particular

$$a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + \cdots + \frac{a_{n-1}+1}{k^{n-1}} \leqslant x$$

which is impossible, since a_{n-1} was chosen to be the largest integer such that the equation above was satisfied. So $a_n \leqslant k-1$. Hence $0 \leqslant a_n \leqslant k-1$ for $i \geqslant 1$.

(b) First, x is clearly an upper bound for $\{r_0, r_1, \dots\}$. Suppose $y < x$. Then we may choose M such that $x - y > \frac{1}{k^M}$. Now consider

$$r_M = a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + \cdots + \frac{a_M}{k^M} \leqslant x.$$

Since a_M is the largest integer for the inequality above is true, we see that $x - r_M \leqslant \frac{1}{k^M}$. This implies that $r_M > y$. So y is not an upper bound for $\{r_0, r_1, \cdots\}$. Therefore since no number less than x is an upper bound for $\{r_0, r_1, \cdots\}$, it follows that $x = \sup\{r_0, r_1, \cdots\}$.

(c) Let $S = \{r_0, r_1, \dots\}$ and $S' = \{r_0', r_1', \dots\}$. Let $x = \sup S = \sup S'$. Suppose that $a_0 \neq a_0'$. Without loss of generality we may assume $a_0 < a_0'$. We observe that

$$\begin{aligned}
r_n &= a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + \cdots + \frac{a_n}{k^n} \\
&\leqslant a_0 + \frac{k-1}{k} + \cdots + \frac{k-1}{k^{n-1}} + \frac{k-1}{k^n} \\
&< a_0 + \frac{k-1}{k} + \cdots + \frac{k-1}{k^{n-1}} + \frac{k}{k^n} \\
&= a_0 + \frac{k-1}{k} + \cdots + \frac{k-1}{k^{n-2}} + \frac{1}{k^{n-1}} \\
&= a_0 + 1
\end{aligned}$$

So in particular $a_0 + 1$ is an upper bound for $\{r_0, r_1, \cdots\}$, implying that $x \leqslant a_0 + 1$. Since $r_0' = a_0'$ is an element of S' and $x = \sup S'$, we see that $a_0' = r_0' \leqslant x \leqslant a_0 + 1$. Since we are assuming $a_0 \neq a_0'$, we must have $a_0' = a_0 + 1$. So in fact $a_0 + 1 = a_0' \leqslant x \leqslant a_0 + 1$, implying that $x = a_0' = a_0 + 1$.

However, recall that by assumption, there is some $i \geqslant 1$ such that $a_i < k-1$. Choose the smallest such i. Then we have

$$r_i = a_0 + \frac{k-1}{k} + \frac{k-1}{k^2} + \cdots + \frac{k-1}{k^{i-1}} + \frac{a_i}{k^i}$$

By the same argument as above, for $n > i$,

$$\begin{aligned}
r_n &< a_0 + \frac{k-1}{k} + \frac{k-1}{k^2} + \cdots + \frac{k-1}{k^{i-1}} + \frac{a_i}{k^i} + \frac{1}{k^i} \\
&= a_0 + \frac{k-1}{k} + \frac{k-1}{k^2} + \cdots + \frac{k-1}{k^{i-1}} + \frac{a_i+1}{k^i} \\
&< a_0 + 1 - \frac{k-(a_i+1)}{k^i}
\end{aligned}$$

This implies that $x = \sup S$ has $x \leqslant a_0 + 1 - \frac{k-(a_i+1)}{k^i} < a_0 + 1$. This is a contradiction. So in fact $a_0 = a_0 + 1$. Repeating the argument proves that $a_i = a_i'$.