MTH 327H: Homework 4

Due: September 28, 2018

1. Office hours the fifth week of classes are M 11:30-12:30, W 3-4, and Th 9-10.
2. Read Rudin Sections 2.15-42.
3. Do problems 6,9,10, and 11 from Rudin Chapter 1.
4. Prove there is exactly one way to give $\mathbb{Q}$ the structure of an ordered field. (That is, given another order $\boxplus$ on $\mathbb{Q}$ that satisfies the axioms of an ordered field, prove that $a \boxplus b$ if and only if $a < b$.)
5. Prove that neither $\mathbb{C}$ nor $\mathbb{Z}/p\mathbb{Z}$ can be given the structure of an ordered field.
6. *Existence and near-uniqueness of decimal expansions in arbitrary base.* Given $x \in \mathbb{R}$ with $x > 0$ and an integer $k \geqslant 2$, define $a_0, a_1, a_2, \ldots$ recursively by setting $a_0 = \lfloor x \rfloor$ (the largest integer less than or equal to x) and a_n to be the largest integer such that

$$a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + \cdots + \frac{a_n}{k^n} \leqslant x.$$

   (a) Show that $0 \leqslant a_i \leqslant k - 1$ for all $i \geqslant 1$.

   (b) Let $r_n = a_0 + \frac{a_1}{k} + \cdots + \frac{a_n}{k^n}$. Show that $\sup\{r_0, r_1, \ldots\} = x$.

   (c) Show that if we have sequences of integers $(a_0, a_1, \ldots)$ and $(a_0', a_1', \cdots)$ such that

   - $0 \leqslant a_i \leqslant k - 1$ and $0 \leqslant a_i' \leqslant k - 1$ for all i.
   - If r_1 and r_i' are the sums defined in part (b) for a_i and a_i' respectively, then $\sup\{r_0, r_1, \ldots\} = \sup\{r_0', r_1', \ldots\}$.
   - For each N there exists $n > N$ and $m > N$ such that $a_n \neq k - 1$ and $a_m' \neq k - 1$.

   then $a_i = a_i'$ for all i.