Homework 3 Solutions

MTH 327H

4. (a) We consider $f\colon [0,1] \to [1,3]$ given by $f(x) = 2x+1$. We claim this is a bijection. For if $g\colon [1,3] \to [0,1]$ is given by $g(x) = \frac{x-1}{2}$, then $g(f(x)) = \frac{(2x+1)-1}{2} = x$ and $f(g(x)) = 2\left(\frac{x-1}{2}\right) + 1 = x$. So g is an inverse to f and f is therefore a bijection.

   (b) Consider the map $f\colon (0,1) \to (0,\infty)$ given by $f(x) = \frac{1}{x} - 1$. We claim this is a bijection. For if $g\colon (0,\infty) \to (0,1)$ is given by $g(x) = \frac{1}{x+1}$, then $g(f(x)) = \frac{1}{(\frac{1}{x}-1)+1} = x$ and $f(g(x)) = \frac{1}{\frac{1}{x+1}} - 1 = x$. Hence g is an inverse to f and f is a bijection.

   (c) Consider the map $f\colon [0,1] \to [0,1)$ defined as follows. Let $f(\frac{1}{2^n}) = \frac{1}{2^{n+1}}$ for $n \in \mathbb{N}$, and $f(x) = x$ for any $x \neq \frac{1}{2^n}$ for some n. (So, eg, $f(1) = \frac{1}{2}$, $f(\frac{1}{2}) = \frac{1}{4}$, and so on. The inverse to this function is $g\colon [0,1) \to [0,1]$ where $g(\frac{1}{2^n}) = \frac{1}{2^{n-1}}$ for $n \in \mathbb{N} - \{0\}$, and $g(x) = x$ otherwise. Since $f \circ g$ and $g \circ f$ are clearly identity maps, f is a bijection.

   (d) Since we constructed a bijection between $\mathbb{R}$ and $(0,1)$ in class, it suffices to show that we can construct a bijection between $(0,1)\times(0,1)$ and $(0,1)$. Let $(a,b) \in (0,1)\times(0,1)$, and let $a = .a_1a_2\cdots$ and $b = .b_1b_2\cdots$ be their decimal expansions. We construct an element of $(0,1)$ as follows. Let $n_1, n_2, n_3, \cdots$ be a list of all of the indices such that $a_{n_i} \neq 9$ and $m_1, m_2, m_3, \cdots$ be a list of all of the indices such that $b_{m_i} \neq 9$. Then we let $f(a,b)$ be $c \in (0,1)$ such that

$$c = .a_1a_2\cdots a_{n_1}b_1\cdots b_{m_1}a_{n_1+1}\cdots a_{n_2}b_{n_1+1}\cdots b_{n_2}\ldots$$

   For example, $f(.3924\ldots, .517\ldots) = .3592147\ldots$. That is, we interleave segments of the decimal expansions of a and b which do not end in 9. The inverse to this function is the map g defined as follows: given $c = .c_1c_2\cdots$, let $k_1, k_2, \cdots$ be a list of all the indices such that $c_{k_i} \neq 9$. Then we let $g(c)$ be the pair (a,b) such that

$$a = .c_1\cdots c_{k_1}c_{k_2+1}\cdots c_{k_3}\cdots$$
$$b = .c_{k_1+1}\cdots c_{k_2}c_{k_3+1}\cdots$$

   For example, $g(.23978\ldots) = (.27\ldots, .398\ldots)$. The purpose of keeping track of segments of decimal expansions not ending in 9 is to prevent g from mapping some c to an a or b ending in a string of trailing 9's (and similarly to prevent non-surjectivity of f). These two functions are clearly inverses, giving the desired bijection.

5. (a) For sequences in $\{0,1\}$, we may repeat Cantor's diagonalization argument. Suppose there is a list of sequences $a_i = (a_{ij})_{j=0}^{\infty}$ such that the list $a_0, a_1, \cdots$ contains every sequence of $0's$ and $1's$. Consider the sequence $b = (b_j)_{j=0}^{\infty}$ defined as follows: for every j, if the jth element a_{ij} of a_i is equal to 0, let $b_j = 1$, and vice versa. Then $b \neq a_j$.

But this is true for arbitrary j, so b is in fact not on this list of sequences! Hence we can make no such list.

(b) Let A be a subset of $\mathbb{N}$. To A I can associate a sequence $f(A)$ in the set S from part (a) as follows. Let $f(A)_i = 1$ if $i \in A$ and $f(A)_i = 0$ if $i \notin A$. This defines a bijection between A and S. Since S is uncountable, A is also uncountable.

(We may also note that binary decimal expansions of real numbers in $(0, 1)$ are a subset of the sequences of elements in $\{0, 1\}$, and note that a set with an uncountable subset is itself uncountable.)