4. (a) We consider \( f : [0, 1] \rightarrow [1, 3] \) given by \( f(x) = 2x + 1 \). We claim this is a bijection. For if \( g : [1, 3] \rightarrow [0, 1] \) is given by \( g(x) = \frac{x - 1}{2} \), then \( g(f(x)) = \frac{(2x+1)-1}{2} = x \) and \( f(g(x)) = 2 \left( \frac{x-1}{2} \right) + 1 = x \). So \( g \) is an inverse to \( f \) and \( f \) is therefore a bijection.

(b) Consider the map \( f : (0, 1) \rightarrow (0, \infty) \) given by \( f(x) = \frac{1}{x} - 1 \). We claim this is a bijection. For if \( g : (0, \infty) \rightarrow (0, 1) \) is given by \( g(x) = \frac{1}{x+1} \), then \( g(f(x)) = \frac{1}{1/(x+1)+1} = x \) and \( f(g(x)) = \frac{1}{x+1} - 1 = x \). Hence \( g \) is an inverse to \( f \) and \( f \) is a bijection.

(c) Consider the map \( f : [0, 1] \rightarrow [0, 1] \) defined as follows. Let \( f(\frac{1}{2^n}) = \frac{1}{2^{n+1}} \) for \( n \in \mathbb{N} \), and \( f(x) = x \) for any \( x \neq \frac{1}{2^n} \) for some \( n \). (So, eg, \( f(1) = \frac{1}{2} \), \( f(\frac{1}{2}) = \frac{1}{4} \), and so on. The inverse to this function is \( g : [0, 1] \rightarrow [0, 1] \) where \( g(\frac{1}{2^n}) = \frac{1}{2^{n+1}} \) for \( n \in \mathbb{N} - \{0\} \), and \( g(x) = x \) otherwise. Since \( f \circ g \) and \( g \circ f \) are clearly identity maps, \( f \) is a bijection.

(d) Since we constructed a bijection between \( \mathbb{R} \) and \((0, 1)\) in class, it suffices to show that we can construct a bijection between \((0, 1) \times (0, 1)\) and \((0, 1)\). Let \((a, b) \in (0, 1) \times (0, 1)\), and let \(a = \cdot a_1 a_2 \cdots\) and \(b = \cdot b_1 b_2 \cdots\) be their decimal expansions. We construct an element of \((0, 1)\) as follows. Let \(n_1, n_2, n_3, \cdots\) be a list of all of the indices such that \(a_{n_i} \neq 9\) and \(m_1, m_2, m_3, \cdots\) be a list of all of the indices such that \(b_{m_i} \neq 9\). Then we let \(f(a, b)\) be \(c \in (0, 1)\) such that

\[
c = \cdot a_1 a_2 \cdots a_{n_1} b_1 \cdots b_{n_2} a_{n_1+1} \cdots a_{n_2} b_{n_1+1} \cdots b_{n_2} \cdots
\]

For example, \(f(.3924\ldots , .517 \ldots ) = .3592147 \ldots\). That is, we interleave segments of the decimal expansions of \(a\) and \(b\) which do not end in 9. The inverse to this function is the map \(g\) defined as follows: given \(c = \cdot c_1 c_2 \cdots\), let \(k_1, k_2, \cdots\) be a list of all the indices such that \(c_{k_i} \neq 9\). Then we let \(g(c)\) be the pair \((a, b)\) such that

\[
a = \cdot c_1 \cdots c_{k_1} c_{k_2+1} \cdots c_{k_3} \cdots
\]
\[
b = \cdot c_{k_1+1} \cdots c_{k_2} c_{k_3+1} \cdots
\]

For example, \(g(.23978 \ldots ) = (.27 \ldots , .398 \ldots )\). The purpose of keeping track of segments of decimal expansions not ending in 9 is to prevent \(g\) from mapping some \(c\) to an \(a\) or \(b\) ending in a string of trailing 9’s (and similarly to prevent non-surjectivity of \(f\)). These two functions are clearly inverses, giving the desired bijection.

5. (a) For sequences in \(\{0, 1\}\), we may repeat Cantor’s diagonalization argument. Suppose there is a list of sequences \(a_i = (a_{ij})_{j=0}^\infty\) such that the list \(a_0, a_1, \cdots\) contains every sequence of 0’s and 1’s. Consider the sequence \(b = (b_j)_{j=0}^\infty\) defined as follows: for every \(j\), if the \(j\)th element \(a_{ij}\) of \(a_i\) is equal to 0, let \(b_j = 1\), and vice versa. Then \(b \neq a_j\).
But this is true for arbitrary $j$, so $b$ is in fact not on this list of sequences! Hence we can make no such list.

(b) Let $A$ be a subset of $\mathbb{N}$. To $A$ I can associate a sequence $f(A)$ in the set $S$ from part (a) as follows. Let $f(A)_i = 1$ if $i \in A$ and $f(A)_i = 0$ if $i \notin A$. This defines a bijection between $A$ and $S$. Since $S$ is uncountable, $A$ is also uncountable.

(We may also note that binary decimal expansions of real numbers in $(0, 1)$ are a subset of the sequences of elements in $\{0, 1\}$, and note that a set with an uncountable subset is itself uncountable.)