Homework 2 Solutions

MTH 327H

3. "Equivalence Relations," Exercise 3. For (a), we must show that $a \sim b$ is an equivalence relation. Let $a, b, c \in \mathbb{R}$.

   Reflexivity: We observe that $\cos a = \cos a$, so $a \sim a$.

   Symmetry: If $a \sim b$, then $\cos a = \cos b$, implying $\cos b = \cos a$, and hence that $b \sim a$.

   Transitivity: Say $a \sim b$ and $b \sim c$. Then $\cos a = \cos b$ and $\cos b = \cos c$, so $\cos a = \cos c$. Hence $a \sim c$.

   For (b), the equivalence class of 0 is $\{a \in \mathbb{R} : a \sim 0\} = \{a \in \mathbb{R} : \cos a = \cos 0 = 1\} = \{2\pi n : n \in \mathbb{Z}\}$. The equivalence class of $\frac{\pi}{2}$ is $\{a \in \mathbb{R} : a \sim \frac{\pi}{2}\} = \{a \in \mathbb{R} : \cos a = \cos \frac{\pi}{2} = 0\} = \{\frac{n\pi}{2} : n \in \mathbb{Z}, n \text{ is odd}\}$.

   "Equivalence Relations," Exercise 9. (a) The order relation $a < b$ on $\mathbb{R}$ is transitive: if $a < b$ and $b < c$, then $a < c$. However, it is not reflexive, for example because it is not true that $1 < 1$, and it is not symmetric, for example because $0 < 1$ does not imply $1 < 0$.

   (b) The subset relation on the power set $P(S)$ of S is reflexive, since for any $A \in P(S)$, we have $A \subset A$. It is also transitive: if $A \subset B$ and $B \subset C$, it follows that $A \subset C$. However, it is not symmetric. For example, if $S = \{1, 2, 3\}$, $A = \{1\}$, and $B = \{1, 2\}$, then $A \subset B$ is true but $B \subset A$ is not true.

   (c) The relation $a \neq b$ on $\mathbb{R}$ is symmetric: if $a \neq b$ then $b \neq a$. However, it is not reflexive, since it is not true that $a \neq a$. Moreover, it is not transitive: we have $3 \neq 7$ and $7 \neq 3$, but it does not follow that $3 \neq 3$.

   (d)The relation $a \sim b$ if $(-1)^a = (-1)^b$ is the same as the relation $a \sim b$ if $2|a - b$ introduced in class. This is reflexive: $a - a = 0$ and $2|0$, so $a \sim a$. It is also symmetric: if $2|a - b$, then $2|b - a$, so $a \sim b$ implies that $b \sim a$. Finally, if $2|a - b$ and $a|b - c$, then $2|(a - b) + (b - c)$, implying that $a|a - c$, so $a \sim b$ and $b \sim c$ together imply that $a \sim c$, and thus $\sim$ is transitive.

   "Induction", Exercise 3. We observe that the described induction fails when $k + 1 = 2$. In this case the set S consists only of the two roses A and B, and removing one only leaves the other; there is no overlap, so the argument does not imply that A and B have the same color.

   (This example is also called, under a slightly different phrasing, the "horse of a different color" problem.)

4. The base case is $n = 1$. In this case the left-hand side of the equation is $\sum_{i=1}^{1} = 1$ and the righthand side is $\frac{(1)^2(1+1)^2}{4} = \frac{4}{4} = 1$, so we conclude that the statement is true for $n = 1$.

For the inductive step, suppose that $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$, and consider $\sum_{i=1}^{n+1} i^3$. We have

$$\begin{aligned}\sum_{i=1}^{n+1} i^3 &= \sum_{i=1}^{n} i^3 + (n+1)^3 \\ &= \frac{n^2(n+1)^2}{4} + (n+1)(n+1)^2 \\ &= \frac{(n^2+4n+4)(n+1)^2}{4} \\ &= \frac{(n+2)^2(n+1)^2}{4} \\ &= \frac{((n+1)+1)^2(n+1)^2}{4}\end{aligned}$$

We conclude that the statement is true for all n.

5. Let $\sim$ be an equivalence relation on S, and let $a, b \in S$. Suppose that $[a] \cap [b] \neq \emptyset$. Then there exists $c \in [a] \cap [b]$. Since $c \in [a]$, $a \sim c$; since $c \in [b]$, $b \sim c$, so by symmetry, $c \sim b$. By transitivity, $a \sim b$. Therefore if $d \in [b]$, we know by definition that $d \sim b$, so $d \sim a$ by transitivity, implying by symmetry that $a \sim d$ and $d \in [a]$. So $[b] \subset [a]$. Similarly $[a] \subset [b]$, and therefore $[a] = [b]$. We conclude that either $[a] \cap [b] = \emptyset$ or $[a] = [b]$. Hence the equivalence classes of $\sim$ partition S.

6. Recall that the operations of addition and multiplication given in class on $\mathbb{N}$ are

$$\begin{cases} a + 0 = a \\ a + S(b) = S(a+b) \end{cases}$$

$$\begin{cases} a \times 0 = 0 \\ a \times S(b) = a + a \times b \end{cases}$$

We first show addition commutes in $\mathbb{N}$. First, we claim that $a + 0 = 0 + a$ for all a. For certainly $0 + 0 = 0$, and inductively if $0 + a = a = a + 0$, then $0 + S(a) = S(0 + a) = S(a + 0) = S(a) = S(a) + 0$. Now, suppose inductively that $a + b = b + a$ for all a. Let us show that $S(b) + a = a + S(b)$ for all a. First, we know that $S(b) + 0 = 0 + S(b)$, by the base case. Suppose we know that $S(b) + a = a + S(b)$ up to some particular a. Then $S(b) + S(a) = S(S(b) + a) = S(a + S(b)) = S(S(a + b)) = S(S(b + a))$, and $S(a) + S(b) = S(S(a) + b) = S(b + S(a)) = S(S(b + a))$. So $S(b)$ commutes with all natural numbers under addition, and therefore addition in $\mathbb{N}$ commutes.

We now show multiplication in $\mathbb{N}$ commutes. First, we claim $0 \times a = 0 = a \times 0$. For certainly $0 \times 0 = 0$. Inductively suppose we know that $0 \times a = 0$, then $0 \times S(a) = 0 + 0 \times a = 0 + 0 = 0$. So 0 commutes with all natural numbers.

Now assume inductively that $a \times b = b\, timesa$ commutes with all $a \in \mathbb{N}$. We know that $S(b) \times 0 = 0 \times S(b)$. Suppose that multiplication by $S(b)$ commutes with all natural numbers up to a particular a, and we know that $a \times S(b) = S(b) \times a$. We then want to show that $S(b) \times S(a) = S(a) \times S(b)$.

$$\begin{aligned} S(b) \times S(a) &= S(b) + S(b) \times a \\ &= S(b) + a \times S(b) \\ &= S(b) + a + a \times b \\ &= a + S(b) + a \times b \\ &= S(a+b) + a \times b \end{aligned}$$

$$\begin{aligned} S(a) \times S(b) &= S(a) + S(a) \times b \\ &= S(a) + b \times S(a) \\ &= S(a) + b + b \times a \\ &= b + S(a) + a \times b \\ &= S(b+a) + a \times b \\ &= S(a+b) + a \times b \end{aligned}$$

Here we have used that multiplication by b commutes with any natural, including $S(a)$, and that multiplication by $S(b)$ commutes with a. We conclude that multiplication is commutative in $\mathbb{N}$.

7. Let S be the set $\{(a,b) : a, b \in \mathbb{Z}, b \neq 0\}$ and $\sim$ be the equivalence relation $(a,b) \sim (c,d)$ if $ad = bc$. Define an operation of multiplication on S by $(a,b) \times (c,d) = (ac, bd)$. In order to show that $\sim$ induces a well-defined operation of multiplication on $\mathbb{Q} = S/\sim$, it suffices to show that if $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$, then $(ac,bd) \sim (a'c', b'd')$. But observe that if $(a,b) \sim (a',b')$, then $ab' = a'b$, and if $(c,d) \sim (c',d')$, then $cd' = c'd$. Multiplying the left and right sides of the equations, we see that $acb'd' = a'c'bd$, implying that $(ac,bd) \sim (a'c', b'd')$ as desired.

8. (a) Reflexivity: For any pair (a,b), we have $a + b = a + b$, so $(a,b) \sim (a,b)$.

   Symmetry: If $(a,b) \sim (c,d)$, then $a + d = c + b$, so since addition commutes in $\mathbb{N}$, $c + b = a + d$, so $(c,d) \sim (a,b)$.

   Transitivity: If $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$, then $a + d = b + c$ and $c + f = d + e$. Adding f to both sides of the first equation gives $a + d + f = b + c + f = b + d + e$. Since $\mathbb{N}$ has additive cancellation, this implies that $a + f = b + e$. So $(a,b) \sim (e,f)$.

   (b) We begin with addition. It suffices to show that if $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$ then $(a+c, b+d) \sim (a'+c', b'+d')$. But the first two equations imply that $a+b' = b+a'$ and $c + d' = c' + d$, so adding these we see that $a + b' + c + d' = b + a' + c' + d$, or $a + c + b' + d' = a' + c' + b + d$. Ergo we conclude that $(a+c, b+d) \sim (a'+c', b'+d')$, implying that this operation of addition is well-defined.

   For multiplication, consider the operation of multiplication on S given by $(a,b) \times (c,d) = (ac+bd, ad+bc)$. First observe that $(a,b) \times (c,d) = (ac+bd, ad+bc) = (ca+$

$db, da + cb) = (c, d) \times (a, b)$, so this operation commutes. Now suppose $(a, b) \sim (a', b')$. We claim that $(a, b) \times (c, d) \sim (a', b') \times (c, d)$. For this is equivalent to $(ac+bd, ad+bc) \sim (a'c + b'd, a'd + b'c) \Leftrightarrow ac + bd + a'd + b'c = a'c + b'd + ad + bc \Leftrightarrow (a + b')c + (b + a')d = (a' + b)c + (a + b')d \Leftrightarrow a + b' = b + a' \Leftrightarrow (a, b) \sim (a', b')$.

These two facts together imply that if $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$, then $(a, b) \times (c, d) \sim (a', b') \times (c, d) \sim (c, d) \times (a', b') \sim (c', d') \times (a', b') \sim (a', b') \times (c', d')$.

(c) Let $[(a, b)], [(c, d)], [(e, f)]$ be elements of $S/\sim$.

(A1) $[(a, b)] + [(c, d)] = [(a + c, b + d)]$ is an element of $S/\sim$.

(A2) $[(a, b)] + [(c, d)] = [(a + c, b + d)] = [(c + a, d + b)] = [(c, d)] + [(a, b)]$.

(A3) $[(a, b)] + ([(c, d)] + [(e, f)]) = [(a, b)] + [(c+e, d+f)] = [(a+(c+e), b+(d+f))] = [((a + c) + e, (b + d) + f)] = [(a + c, b + d)] + [(e, f)] = ([(a, b)] + [(c, d)]) + ([b + f])$.

(A4) Consider the element $[(0, 0)]$. We see that $[(a, b)]+[(0, 0)] = [(a+0, b+0)] = [(a, b)]$ for any $[(a, b)]$. So $[(0, 0)]$ is the additive identity.

(A5) For any $[(a, b)]$, let $-[(a, b)] = [(b, a)]$. Then $[(a, b)] + -[(a, b)] = [(a, b)] + [(b, a)] = [(a + b, b + a)] = [(0, 0)]$, since $(a + b) + 0 = 0 + (b + a)$.

(M1) $[(a, b)] \times [(c, d)] = [(ac + bd, ad + bc)]$ is an element of $S/\sim$.

(M2) $[(a, b)] \times [(c, d)] = [(ac + bd, ad + bc)] = [(db + ca, cb + da)] = [(c, d)] + [(b, a)]$.

(M3) We observe that

$$\begin{aligned}[(a, b)] \times ([(c, d)] \times [(e, f)]) &= [(a, b)] \times [(ce + df, cf + de)] \\ &= [(a(ce + df) + b(cf + de)), a(cf + de) + b(ce + df))] \\ &= [(ace + adf + bcf + bde, acf + ade + bce + bdf)]\end{aligned}$$

and

$$\begin{aligned}([(a, b)] \times [(c, d)]) \times [(e, f)] &= ([ac + bd, ad + bc)] \times [(e, f)] \\ &= [(ac + bd)e + (ad + bc)f, (ac + bd)f + (ad + bc)e)] \\ &= [(ace + bde + adf + bcf, acf + bdf + ade + bce)].\end{aligned}$$

Ergo $[(a, b)] \times ([(c, d)] \times [(e, f)]) = ([(a, b)] \times [(c, d)]) \times [(e, f)]$.

(M4) Consider the element $[(1, 0)]$. Then $[(a, b)] \times [(1, 0)] = [(a(1)+b(0), a(0)+b(1))] = [(a, b)]$. Moreover $[(1, 0)] \neq [(0, 0)]$, since $1+0 \neq 0+0$. Ergo $[(1, 0)]$ is the multiplicative identity.

(D) Observe that

$$\begin{aligned}[(a, b)] \times ([(c, d)] + [(e, f)]) &= [(a, b)] \times [(c + e, d + f)] \\ &= [(a(c + e) + b(d + f), a(d + f) + b(c + e))] \\ &= [((ac + bd) + (ae + bf), (ad + bc, af + be))] \\ &= [(ac + bd, ad + bc)] + [(ae + bf, af + be)] \\ &= [(a, b)] \times [(c, d)] + [(a, b)] \times [(d, e)].\end{aligned}$$

However, notice that (M5) is false. For suppose that $[(2,0)]$ has a multiplicative inverse. This implies that there exists $[(a,b)]$ such that $[(1,0)] = [(2,0)] \times [(a,b)] = [(2a,2b)]$. In particular, $(1,0) \sim (2a,2b)$, so $1+2b = 2a+0$. This is impossible, since one side of the equation is even and one side is odd.

9. We will construct a function H between $A^{B\times C}$ and $(A^B)^C$. Let $f \in A^{B\times C}$. Then $f\colon B\times C \to A$ maps a pair $(b,c) \mapsto f(b,c)$. Let $H(f) \in (A^B)^C$ be the function $H(f)\colon C \to A^B$ such that $H(f)(c)\colon B \to A$ maps b to $f(b,c)$.

   In order to show that H is a bijection, we construct its inverse G. Let $j\colon C \to A^B$, so that $j(c)$ is a function $B \to A$. Then let $G(j)\colon B\times C \to A$ be the function $G(j)(b,c) = j(c)(b)$.

   Let us show that $H \circ G$ and $G \circ H$ are identity maps. First, let $j\colon C \to A^B$, and let $G(j)(b,c) = j(c)(b)$. Then $H(G(j))$ is the function mapping c to $j(c)\colon B \to A$; this means that $H(G(j))$ is exactly the function j. Similarly, if $f \in A^{B\times C}$, we have $H(f)$ is the function mapping c to $H(f)(c)\colon B \to A$ where $b \mapsto f(b,c)$. It then follows that $G(H(f))\colon B\times C \to A$ is the function mapping $(b,c) \to f(b,c)$. We conclude that G is a bijection.

10. Suppose that A and B are nonempty finite subsets. Let $A = \{x_1, \cdots, x_m\}$, with $|A| = m$, and $B = \{y_1, \cdots, y_n\}$, with $|B| = n$. We can give a complete description of a function $f\colon B \to A$ by choosing an element $f(y_i) \in A$ for each $1 \le i \le n$. There are m possibilities for each $f(y_i)$, giving m^n functions total. So there are $m^n = |A|^{|B|}$ functions from B to A. So $|A^B| = |A|^{|B|}$ as desired.