Homework 1 Solutions

MTH 327H

6. Let S, T, and V be sets. First we will show that $S \cup (T \cap V)$ is a subset of $(S \cup T) \cap (S \cup V)$. Suppose that $x \in S \cup (T \cap V)$. Then one of $x \in S$ or $x \in T \cap V$ is true. In the first case, if $x \in S$, it follows that $x \in S \cup T$ and $x \in S \cup V$. Therefore since x is in both of these sets, we see that $x \in (S \cup T) \cap (S \cup V)$. In the second case, if x is in $T \cap V$, we see that $x \in T$ and $x \in V$. Therefore it is also true that $x \in S \cup V$ and $x \in S \cup T$. Since both of these things are true, it follows that $x \in (S \cup V) \cap (S \cup T)$. Since x was an arbitrary element of $S \cup (T \cap V)$, we conclude that $S \cup (T \cap V) \subseteq (S \cup T) \cap (S \cup V)$.

   Now we will show that $(S \cup T) \cap (S \cup V)$ is a subset of $S \cup (T \cap V)$. Let $x \in (S \cup T) \cap (S \cup V)$. Then $x \in S \cup T$ and $x \in S \cup V$. There are two cases, $x \in S$ and $x \notin S$. If $x \in S$, then $x \in S \cup (T \cap V)$. If $x \notin S$, then since $x \in S \cup T$, we must have $x \in T$, and since $x \in S \cup V$, we must have $x \in V$. Ergo since $x \in T$ and $x \in V$, we see that $x \in T \cap V$. Therefore $x \in S \cup (T \cap V)$. Since in either case $x \in S \cup (T \cap V)$, we conclude that $(S \cup T) \cap (S \cup V) \subseteq S \cup (T \cap V)$.

   Therefore, since $S \cup (T \cap V) \subseteq (S \cup T) \cap (S \cup V)$ and $(S \cup T) \cap (S \cup V) \subseteq S \cup (T \cap V)$ are both true, it must be the case that $S \cup (T \cap V) = (S \cup T) \cap (S \cup V)$.

7. Sets and Logic Exercises

   (15) Let B and C be nonempty sets, and $f \colon B \times C \to C \times B$ be the function given by $f(x, y) = (y, x)$. I claim this function is a bijection. By a lemma in class, it suffices to show that f has an inverse. Let g be the function $g \colon C \times B \to B \times C$ be the function given by $g(y, x) = (x, y)$. Then $g \circ f(x, y) = g(y, x) = (x, y)$ and $f \circ g(y, x) = f(x, y) = (y, x)$. We conclude that g is an inverse function for f. Therefore f is a bijection.

   (19) Let (x, y) be an $A \times (B \cup C)$, so that (by definition) $x \in A$ and $y \in B \cup C$. Then either $y \in B$ or $y \in C$. In the first case, $(x, y) \in A \times B$, and in the second case $(x, y) \in A \times C$. Either of these possibilities implies that $(x, y) \in (A \times B) \cup (A \times C)$. Since (x, y) was an arbitrary element of $A \times (B \cup C)$, we conclude that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

   Conversely, suppose $(x, y) \in (A \times B) \cup (A \times C)$. Then either $(x, y) \in A \times B$ or $(x, y) \in A \times C$. If $(x, y) \in A \times B$, then $x \in A$ and $y \in B$. The second statement implies that $y \in B \cup C$. So (x, y) is an element of $A \times (B \cup C)$. If instead $(x, y) \in A \times C$, a similar chain of reasoning shows $(x, y) \in A \times (B \cup C)$. Since (x, y) was an arbitrary element of $(A \times B) \cup (A \times C)$, we conclude that $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.

   Since both $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ and $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$ are true, we conclude that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

(20) (a) We want to show $U-(A\cap B)=(U-A)\cup(U-B)$. First let $x\in U-(A\cap B)$. Then $x\in U$ but $x\notin A\cap B$. Since x is not in $A\cap B$, either $x\notin A$ or $x\notin B$. If $x\notin A$, then $x\in U-A$. So $x\in(U-A)\cup(U-B)$. Similarly, if $x\notin B$, then $x\in(U-A)\cup(U-B)$. So $U-(A\cap B)\subseteq(U-A)\cup(U-B)$.

Conversely, let $x\in(U-A)\cup(U-B)$. Then either $x\in U-A$ or $x\in U-B$. If $x\in U-A$, then $x\notin A$, so $x\notin A\cap B$. Hence $x\in U-(A\cap B)$. Similarly if $x\in U-B$, it follows that $x\in U-(A\cap B)$. Hence we conclude that $U-(A\cap B)=(U-A)\cup(U-B)$.

(b) We want to show $U-(A\cup B)=(U-A)\cap(U-B)$. First, we claim that if S is any subset of U, $U-(U-S)=S)$. For if $x\in S$, then $x\notin U-S$, so $x\in U-(U-S))$; the converse argument is identical. Using this, we see $A\cup B=(U-(U-A))\cup(U-U-B))=U-((U-A)\cap(U-B))$ by part (a). But then $U-(A\cup B)=U-(U-((U-A)\cap(U-B)))=(U-A)\cap(U-B)$.

(22) No; consider $A=\{1,2,3\}$ and $B=\{2,3,4\}$. We see that $|A|=3=|B|$, but $|A\cup B|=4$.

(28) (a) Let $f\colon B\to C$ and $g\colon C\to D$ such that $g\circ f\colon B\to D$ is injective. Let x_1,x_2 be any two elements of B such that $f(x_1)=f(x_2)$. Then $g\circ f(x_1)=g\circ f(x_2)$. Since $g\circ f$ is injective, it follows that $x_1=x_2$. Hence f is injective.

(b) Let f be the function $f\colon\{0\}\to\{1,2\}$ such that $f(0)=1$ and let g be the function $g\colon\{1,2\}\to\{3\}$ such that $g(1)=g(2)=3$. Then g is not injective, since $g(1)=g(2)$, but $g\circ f\colon\{0\}\to\{3\}$ is the function given by $g\circ f(0)=3$, hence is injective.

(29) (a) Let $f\colon B\to C$ and $g\colon C\to D$ such that $g\circ f\colon B\to D$ is surjective. Let $z\in D$. Then by assumption, there exists $x\in B$ such that $g\circ f(x)=z$. But then if $y=f(x)$, we see that in fact $g(y)=z$. So for any $z\in D$ there exists a $y\in C$ such that $g(y)=z$. Ergo g is surjective.

(b) Let f be the function $f\colon\{0\}\to\{1,2\}$ such that $f(0)=1$ and let g be the function $g\colon\{1,2\}\to\{3\}$ such that $g(1)=g(2)=3$. Then f is not surjective, since there is no $x\in\{0\}$ such that $f(x)=2$, but $g\circ f\colon\{0\}\to\{3\}$ is the function given by $g\circ f(0)=3$, hence is surjective.

(33)(a) Let $f\colon B\to C$ and $S,T\subseteq B$. Let $y\in f(S\cup T)$. By definition, there exists $x\in S\cup T$ such that $f(x)=y$. Since $x\in S\cup T$, either $x\in S$ or $x\in T$. If $x\in S$, $f(x)=y\in f(S)$. If instead $x\in T$, $f(x)=y\in f(T)$. In either case, we see $y\in f(S)\cup f(T)$. So since y was arbitrary, $f(S\cup T)\subseteq f(S)\cup f(T)$.

Conversely, suppose $y\in f(S)\cup f(T)$. Then either $y\in f(S)$ or $y\in f(T)$. If $y\in f(S)$, there exists $x\in S$ such that $f(x)=y$. Since x is an element of S, x is also an element of $S\cup T$, so $y\in f(S\cup T)$. Likewise if $y\in f(T)$ it follows that $y\in f(S\cup T)$.

It follows that $f(S\cup T)=f(S)\cup f(T)$.

(c) Consider $f\colon \{1,2\} \to 3$ given by $f(1) = 3$ and $f(2) = 3$. Then if $S = \{1\}$ and $T = \{2\}$, we see that $f(S \cap T) = f(\phi) = \phi$ but $f(S) \cap f(T) = \{3\} \cap \{3\} = \{3\}$.

(34) Let $f\colon B \to C$ be a function. We claim that f is injective if and only if $f(S \cap T) = f(S) \cap f(T)$ for every pair of subsets S and T of B.

For the if direction, suppose that f is injective. Recall from question (33)b, which we did in class, that for any S and T subsets of B, we know that $f(S \cap T) \subset f(S) \cap f(T)$. Now let $y \in f(S) \cap f(T)$. This implies that $y \in f(S)$ and $y \in f(T)$. Then since $y \in f(S)$, there must be an element $x_1 \in S$ such that $f(x_1) = y$, and since $y \in f(T)$, there must be an element $x_2 \in T$ such that $f(x_2) = y$. I claim $x_1 \neq x_2$. But then $f(x_1) = f(x_2)$, so since f is injective $x_1 = x_2$, and therefore $x_1 \in S$ and $x_1 = x_2 \in T$ are both true. This implies that $x_1 \in S \cap T$, and therefore $f(x_1) = y \in f(S \cap T)$. Since y was an arbitrary element of $f(S) \cap f(T)$, we see that $f(S) \cap f(T) \subseteq f(S \cap T)$. Therefore since we have inclusions in both directions, we see that $f(S \cap T) = f(S) \cap f(T)$ if f is injective.

Conversely, suppose that $f(S \cap T) = f(S) \cap f(T)$ for every pair of subsets S and T of B. Then let x_1 and x_2 be any two elements of B such that $f(x_1) = y = f(x_2)$. Let $S = \{x_1\}$ and $T = \{x_2\}$. Notice that $S \cap T$ is empty if $x_1 \neq x_2$ and equal to $\{x_1\}$ if $x_1 = x_2$. Now by assumption we observe that $f(S \cap T) = f(S) \cap f(T) = \{f(x_1)\} \cap \{f(x_2)\} = \{y\} \cap \{y\} = \{y\}$. This implies that $f(S \cap T) = \{y\}$ is nonempty, and therefore $S \cap T$ is nonempty. Hence $x_1 = x_2$. Since x_1 and x_2 were arbitrary elements of B with $f(x_1) = f(x_2)$, we conclude that f is injective.