Exam 2 Solutions

MTH 327H

1. (a) Let $\epsilon > 0$ and let P be a partition of $[a, b]$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$. We may insist that $c = x_j$ for some $1 < j < n$ by taking a refinement if necessary (this can only decrease the difference $U(P, f, \alpha) - L(P, f, \alpha)$). Let $P_1 = \{a = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_j = c\}$ and $P_2 = \{c = x_j \leqslant x_{j+1} \leqslant \cdots \leqslant x_n = b\}$. Then it follows immediately that

$$U(P, f, \alpha) = U(P_1, f, \alpha) + U(P_2, f, \alpha)$$
$$L(P, f, \alpha) = L(P_1, f, \alpha) + L(P_2, f, \alpha)$$

We see that $U(P_i, f, \alpha) - L(P_i, f, \alpha) < \epsilon$ for $i = 1, 2$. Ergo f is integrable with respect to α. Furthermore we see that

$$\begin{aligned}\int_a^b f d\alpha &< U(P, f, \alpha) \\ &= U(P_1, f, \alpha) + U(P_2, f, \alpha) \\ &\leqslant \int_a^c f d\alpha + \int_c^b f d\alpha + \epsilon\end{aligned}$$

Since ϵ was arbitary, we have $\int_a^b f d\alpha \leqslant \int_a^c f d\alpha + \int_c^b f d\alpha$. Repeating the argument using lower sums, we conclude that in fact

$$\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha.$$

(b) Let α and β be monotone increasing functions on $[a, b]$. Let $P = \{a = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n = b\}$ be any partition of $[a, b]$. Then

$$U(P, \alpha, \beta) = \sum_{i=1}^{n} \alpha(x_i)[\beta(x_i) - \beta(x_{i-1})]$$
$$L(P, \alpha, \beta) = \sum_{i=1}^{n} \alpha(x_{i-1})[\beta(x_i) - \beta(x_{i-1})]$$

so we see that

$$U(P, \alpha, \beta) - L(P, \alpha, \beta) = \sum_{i=1}^{n} [\alpha(x_i) - \alpha(x_{i-1})][\beta(x_i) - \beta(x_{i-1})].$$

Observe that this expression is symmetric in α and β, and therefore also equals $U(P, \beta, \alpha) - L(P, \beta, \alpha)$. Therefore given $\epsilon > 0$, it is possible to choose P such that $U(P, \alpha, \beta) - L(P, \alpha, \beta) < \epsilon$ if and only if it is possible to choose P such that $U(P, \beta, \alpha) - L(P, \beta, \alpha) < \epsilon$. Ergo $\alpha \in \mathcal{R}(\beta)$ if and only if $\beta \in \mathcal{R}(\alpha)$.

2. (a) We use the change of variable formula. Let $s = \phi(x) = x^3$, so that $\phi'(x) = 3x^2$. Then the change of variable formula implies that

$$\int_0^1 f(x)x^2 dx = \frac{1}{3}\int_0^1 f(s^{\frac{1}{3}})ds$$

Now consider the function $F(t) = \frac{1}{3}\int_0^t f(s^{\frac{1}{3}})ds$. By the Fundamental Theorem, $F'(t) = \frac{1}{3}f(s^{\frac{1}{3}})$. Then the Mean Value Theorem implies that there exists $b \in (0,1)$ such that

$$F(1) - F(0) = \frac{1}{3}f(b^{\frac{1}{3}})$$

But $F(0) = 0$ and $F(1) = \frac{1}{3}\int_0^1 f(s^{\frac{1}{3}})ds = \int_0^1 f(x)x^2 dx$. So if we let $a = b^{\frac{1}{3}}$, we see that there is some $a \in (0,1)$ for which

$$\int_0^1 f(x)x^2 dx = \frac{1}{3}f(a).$$

(b) Observe that since each derivative $f^{(m)}$ is differentiable on $[0,1]$, it is in particular continuous on $[0,1]$. Let $U_m = (f^{(m)})^{-1}(\mathbb{R} - \{0\})$. By continuity, U_i is open; by assumption, the open sets U_i cover $[0,1]$. Since $[0,1]$ is compact, this implies some finite collection $U_1, U_1, \ldots, U_M$ cover $[0,1]$. Hence for any $x \in [0,1]$ there is some $m \leqslant M$ for which $f^{(m)}(x) \neq 0$.

3. (a) We observe that the condition on f trivially implies that f is continuous (indeed, uniformly continuous).

First, if $\alpha = 0$ then $f\colon X \to X$ is constant with $f(X) = \{p\}$ for some point $p \in X$, and therefore f fixes exactly p.

Now assume $\alpha \in (0,1)$. Observe that f has at most one fixed point: if $f(p) = p$ and $f(q) = q$, then $d(f(p), f(q)) = d(p,q) > \alpha d(p,q)$. So it suffices to find a single fixed point of f.

Choose any $x \in X$. Consider the sequence $(x, f(x), f^2(x), \ldots)$ with $s_0 = x$ and $s_n = f^n(x)$. Let $\beta = d(x, f(x))$, such that $d(f^n(x), f^{n+1}(x)) \leqslant \alpha^n \beta$. I claim that $\{s_n\}$ is a Cauchy sequence. For given $\epsilon > 0$, I may choose N such that $\epsilon > \alpha^N \cdot \frac{\beta}{1-\alpha}$. Then given $n < m > N$, I have

$$
\begin{aligned}
d(f^n(x), f^m(x)) &\leqslant \sum_{i=n}^{m-1} d(f^i(x), f^{i+1}(x)) \\
&\leqslant \sum_{i=n}^{m-1} \alpha^i \beta \\
&\leqslant \alpha^n \beta \sum_{i=0}^{m-n-1} \alpha^i \\
&\leqslant \alpha^N \beta \cdot \frac{1}{1-\alpha} \\
&< \epsilon
\end{aligned}
$$

But X is complete, so Cauchy sequences converge. Hence $s_n \to p$ for some p. But $\{f(s_n)\}$ is the same sequence with s_0 deleted, and must also converge to p. By continuity, $f(p) = p$.

(b) For example $f(x) = \arctan x - 1$ on $[-1, \infty)$. We observe that the derivative of the arctangent function is $f'(x) = \frac{1}{1+x^2}$, hence $0 < f'(x) < 1$ for all x. Therefore given $a < b \in [-1, \infty)$, by the Mean Value Theorem we have $f(b) - f(a) = f'(x)(b-a)$ for some $x \in (a, b)$, hence $|f(b) - f(a)| < |b - a|$. But if $g(x) = x - (\arctan x - 1)$, we see $g(-1) = \frac{\pi}{4} > 0$ and $g'(x) = 1 - \frac{1}{1+x^2} > 0$, so g is increasing and therefore nonzero on $[-1, \infty)$. Ergo f has no fixed point on $[-1, \infty)$.

(c) We observe that g is clearly continuous. Consider the function $h\colon X \to R$ given by $d(p, h(p))$. This is a continuous real-valued function off a compact set. Hence h attains its bounds. Let $a = \inf h(x) \geqslant 0$. Then there is some p such that $h(p) = a$, that is, such that $d(p, g(p)) = a$. But then if $a \neq 0$, $d(g(p), g^2(p)) < a$ by assumption. This is a contradiction. So $d(p, g(p)) = 0$, and p is a fixed point of g.

(d) Consider $f\colon [0, \frac{1}{2}] \to [0, \frac{1}{2}]$ given by $f(x) = x - x^2$. If $0 \leqslant x < y \leqslant \frac{1}{2}$, we have

$$
\begin{aligned}
f(y) - f(x) &= (y - y^2) - (x - x^2) \\
&= (y - x) - (y - x)(y + x) \\
&= (1 - y - x)(y - x) \\
&< y - x
\end{aligned}
$$

4. (a) The number of n-digit positive integers containing no 0 is 9^n, and the reciprocal of each such integer is no more than $\frac{1}{10^{n-1}}$. Therefore the sum of this series is less than or equal to

$$
\sum_{n=1}^{\infty} \frac{9^n}{10^{n-1}} = 9 \sum_{n=0}^{\infty} \left(\frac{9}{10}\right)^n = \frac{9}{1 - \frac{9}{10}} = 90
$$

(b) Suppose $\sum b_n$ converges. Choose a real number $d > c$. Then there exists N such that $n \geqslant N$ implies that $\frac{a_n}{b_n} < d$, so in particular $0 < a_n < db_n$. Now db_n

converges, so by the Comparison Test a_n converges. Similarly, since $\lim \frac{b_n}{a_n} = \frac{1}{c}$ is also a positive real, convergence of $\sum a_n$ implies convergence of $\sum b_n$. We conclude that $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

5. (a) First, observe that the set of real continuous functions on $\mathbb{R}$ is certainly not countable; the constant functions form a subset having the cardinality of $\mathbb{R}$. Now recall that a continuous function on $\mathbb{R}$ is determined by the values it takes on $\mathbb{Q}$. Indeed, if $\{r_1, r_2, r_3, \dots\}$ is an enumeration of the rationals, we may write out the decimal expansions of $\{f(r_1), f(r_2), f(r_3), \dots\}$ and collect terms along the diagonals to show f is determined by a single real number. (Since we are presently trying to prove that the cardinality of the set of real continuous functions on $\mathbb{R}$ is no more than the cardinality of $\mathbb{R}$, possible duplication is irrelevant here.) We conclude that the set of real continuous functions has the same cardinality as $\mathbb{R}$.

   (b) No; consider the function $f(x) = \frac{1}{x-\sqrt{2}}$.

6. (a) This is impossible; $[0,1]$ is compact, and therefore its image under a continuous function is compact, but $(0,1)$ is not compact.

   (b) We may use the function

$$f(x) = \begin{cases} 0 & 0 < x \leqslant \frac{1}{4} \\ 2x - \frac{1}{2} & \frac{1}{4} \leqslant x \leqslant \frac{3}{4} \\ 1 & \frac{3}{4} < x < 1 \end{cases}$$

   (c) This is impossible.

   Lemma: An injective continuous function $f\colon (a,b) \to \mathbb{R}$ is necessarily strictly monotone.

   Proof: Suppose not. Then we may find three points $x < y < z$ in (a,b) such that either $f(x) < f(y)$ and $f(y) > f(z)$ or $f(x) > f(y)$ and $f(y) < f(z)$. (These are all strict inequalities because of injectivity.) Focusing on the first case, we may choose a real number c such that $f(x), f(z) < c < f(y)$. Then the Intermediate Value Theorem implies that there is some $w_1 \in (x,y)$ such that $f(w_1) = c$ and some $w_2 \in (y,z)$ such that $f(w_2) = c$. This contradicts injectivity. So f is strictly monotone.

   Now, suppose there exists a continuous bijection $f\colon [0,1] \to (0,1)$. Then it must be strictly monotone. But from class, the inverse of a continuous strictly monotone function is also continuous, so f^{-1} is a continuous map from $(0,1)$ onto $[0,1]$. As observed in part (a), this is impossible because of the compactness of $[0,1]$. so no such f exists.

7. (a) Positivity and symmetry are clear. For the triangle inequality, given $f, g, h\colon [0,1] \to \mathbb{R}$, we have

$$\begin{aligned}\int_0^1 |f-g|dx &\leqslant \int_0^1 (|f-h|+|h-g|)dx \\ &= \int_0^1 |f-h|dx + \int_0^1 |h-g|dx\end{aligned}$$

as desired.

(b) Consider the sequence of functions $f_n\colon [0,1] \to \mathbb{R}$ for $n \geqslant 2$ defined by

$$f_n(x) = \begin{cases} n^3x & 0 \leqslant x \leqslant \frac{1}{n^2} \\ 2n - n^3x & \frac{1}{n^2} < x \leqslant \frac{2}{n^2} \\ 0 & x > \frac{2}{n^2} \end{cases}$$

Each function f_n is continuous, ergo integrable, and $\int_0^1 |f_n|dx = \int_0^1 f_n dx = \frac{1}{n}$. Ergo in the ℓ^1 metric $\{f_n\}$ converges to the zero function. However, in the ℓ^∞ metric, the sequence $\{f_n\}$ is unbounded, ergo not convergent.

The other direction is impossible: We have $\ell^1(f,g) = \int_0^1 |f-g|dx \leqslant sup_{x\in[0,1]}|f-g|(1-0) = \ell^{infty}(f,g)$. So any sequence that converges with respect to ℓ^∞ also converges with respect to ℓ^1.

8. (a) Consider the function $j(x) = f(1-x)$, which is clearly infinitely differentiable and nonzero exactly on $(-\infty, 1)$. The function $g(x) = f(x)j(x)$ then satisfies the indicated conditions is then infinitely differentiable and nonzero on exactly $(0,1)$.

(b) Consider the function $h(x) = \frac{f(x)}{f(x)+j(x)}$, where $j(x)$ is as in part (a). We observe that $f(x) + j(x) \neq 0$ on $\mathbb{R}$, so $h(x)$ is differentiable and indeed infinitely differentiable. On $x \leqslant 0$, we have $h(x) = \frac{0}{j(x)} = 0$, and on $x \geqslant 1$, we have $h(x) = \frac{f(x)}{0+j(x)}$.

This example, or rather its generalization to $\mathbb{R}^n$, is one of the foundational tools of differential geometry.