MTH 327H: Exam 2

Due Date: Friday, December 7, at 10:20 a.m.

Instructions: Please submit complete solutions to **exactly five** of the questions below. If you submit more than five solutions, the first five on the paper you hand in will be graded. Each problem is worth ten points. Problems are **not in order of difficulty.** Do not use any resources other than the textbook and your notes and homework to work on the exam. Do not discuss the problems with anyone (other than me). If you spot anything you are concerned is a potential typo, please email me for confirmation or correction immediately.

1. Carefully prove the following properties of Riemann-Stieltjes integration.

   (a) If $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and $a < c < b$, then $f \in \mathcal{R}(\alpha)$ on $[a, c]$ and $[c, b]$, and

   $$\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha.$$

   (b) If α and β are monotone increasing functions on $[a, b]$, then $\alpha \in \mathcal{R}(\beta)$ if and only if $\beta \in \mathcal{R}(\alpha)$.

2. Prove the following statements.

   (a) Let f be a continuous real-valued function. Show that $\int_0^1 f(x)x^2 dx = \frac{1}{3}f(a)$ for some $a \in [0, 1]$.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function, and suppose that for each $x \in [0, 1]$, there exists some m such that $f^{(m)}(x) \neq 0$. Prove that the following stronger statement holds: There exists M such that for each $x \in [0, 1]$, there exists $m \leqslant M$ such that $f^{(m)}(x) \neq 0$.

3. Let (X, d) be a metric space. A function $f : X \to X$ is a *contraction* if there exists $\alpha \in [0, 1)$ such that

   $$d(f(x), f(y)) \leqslant \alpha d(x, y) \text{ for all } x \neq y \in X$$

   A point p is a *fixed point* of a function f if $f(p) = p$.

   (a) Prove that every contraction of a complete metric space has a unique fixed point.

   (b) Let $g : X \to X$ satisfy

   $$(*) \qquad d(g(x), g(y)) < d(x, y) \text{ for all } x \neq y \in X.$$

   Show by example that g may fail to have fixed points, even for complete X.

   (c) Show that if X is compact, a function $g : X \to X$ which satisfies $(*)$ has a unique fixed point.

   (d) Give an example of a compact metric space X and a function $g : X \to X$ satisfying $(*)$ such that g is not a contraction.

4. (a) Consider the series $\sum_{n=1}^{\infty} s_n$ such that $s_n = \frac{1}{n}$ if the decimal expression of n does not contain 0, and $s_n = 0$ otherwise. (So, eg, $s_{17} = \frac{1}{17}$ but $s_{107} = 0$.) Prove that $\sum s_n$ converges to a number less than 90.

(b) Let $\sum a_n$ and $\sum b_n$ be series of positive numbers. Suppose that $\lim \frac{a_n}{b_n} = c$ for some $0 < c < \infty$. Prove that either both series converge or both series diverge. (This slightly extends the Comparison Test.)

5. (a) Prove that the set of continuous real functions on the real line has the cardinality of the reals. (Hint: Recall that any continuous function on a metric space X is determined by its values on a dense subset of X.)

(b) Is it the case that any continuous function $g\colon \mathbb{Q} \to \mathbb{R}$ can be extended to a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f|_{\mathbb{Q}} = g$?

6. For each of the following, either give an example of a function with the indicated property, or prove that you can't.

(a) A surjective continuous function $f\colon [0,1] \to (0,1)$.

(b) A surjective continuous function $f\colon (0,1) \to [0,1]$.

(c) A *bijective* continuous function $f\colon (0,1) \to [0,1]$.

7. Let $\mathcal{B}$ be the set of continuous functions $f\colon [0,1] \to \mathbb{R}$. We know that one possible metric on this space is

$$\ell^\infty(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$$

Another metric on this space is

$$\ell^1(f,g) = \int_0^1 |f(x) - g(x)| dx$$

(a) Use the properties of Riemann integration to prove that this is in fact a metric.

(b) Exhibit a sequence in $\mathcal{B}$ that converges with respect to the ℓ^1 metric but not the ℓ^∞ metric. Is it possible to do the opposite?

8. Recall that in class we showed that the function

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leqslant 0 \end{cases}$$

is infinitely differentiable with $f^{(n)}(0) = 0$ for all n. [If you weren't there, you should satisfy yourself that this is true at some point, although you don't need that argument for the problem below.]

(a) Produce an infinitely differentiable function $g(x)$ which is nonzero on $(0,1)$ and zero elsewhere.

(b) Produce an infinitely differentiable function $h(x)$ such that $h(x) = 0$ for $x \leqslant 0$ and $h(x) = 1$ for $x \geqslant 1$.