Exam 1 Solutions

MTH 327H

1. (a) For any (x, y) in $\mathbb{R}^2$, we see that

$$\begin{aligned} d_1((x,y),(x',y'))^2 &= (x-x')^2 + (y-y')^2 \\ &\leqslant (x-x')^2 + 2|x-x'||y-y'| + (y-y')^2 \\ &= (|x-x'| + |y-y'|)^2 \\ &= d_2((x,y),(x',y')), \end{aligned}$$

so $d_1((x,y),(x',y')) \leqslant d_2((x,y),(x',y'))$. Moreover,

$$\begin{aligned} d_3((x,y),(x',y'))^2 &= \sup\{(x-x')^2, (y-y')^2\} \\ &\leqslant (x-x')^2 + (y-y')^2 \\ &= d_1((x,y),(x',y'))^2, \end{aligned}$$

so

$$d_3((x,y),(x',y')) \leqslant d_1((x,y),(x',y')) \leqslant d_2((x,y),(x',y')).$$

Now observe that

$$\begin{aligned} d_1((x,y),(x',y')) &= \sqrt{(x-x')^2 + (y-y')^2} \\ &\leqslant \sqrt{2(\sup\{|x-x'|, |y-y'|\})^2} \\ &= \sqrt{2} d_3((x,y),(x',y')). \end{aligned}$$

Moreover clearly $d_2((x,y),(x',y')) \leqslant 2d_3((x,y),(x',y'))$. So in general we have

$$\begin{aligned} d_3((x,y),(x',y')) \leqslant d_1((x,y),(x',y')) \leqslant \sqrt{2} d_3((x,y),(x',y')) \\ d_3((x,y),(x',y')) \leqslant d_2((x,y),(x',y')) \leqslant 2d_3((x,y),(x',y')) \\ d_1((x,y),(x',y')) \leqslant d_2((x,y),(x',y')) \leqslant 2d_1((x,y),(x',y')) \end{aligned}$$

(b)If we have two metrics d and d' on a space X, in order to show that if a set U is open in (X, d) then it is open in (X, d'), it suffices to show that any neighborhood $N_r^d(p) = \{q \in X : d(p,q) < r\}$ contains a neighborhood $N_{r'}^{d'}(p) = \{q \in X : d(p,q) < r'\}$. For then if $U \subset X$ and p is an interior point of U with respect to the metric (X, d), then p is also an interior point of U with respect to the metric d'.

So if there exist $0 < \alpha < \beta$ such that $\alpha d(p,q) \leqslant d'(p,q) \leqslant \beta d(p,q)$, this implies that for $r \in \mathbb{R}$, we have $N_{\alpha r}^{d'}(p) \subset N_r^d(p)$ and $N_{\frac{r}{\beta}}^{d}(p) \subset N_r^{d'}(p)$. Hence strongly equivalent metrics induce the same topology.

2. (a) Since we already know $\mathbb{R}$ is a field, it suffices to check that $\mathbb{Q}(\sqrt{2})$ is closed under addition and multiplication, is closed under taking additive and multiplicative inverses, and contains 0 and 1. Clearly $0, 1 \in \mathbb{Q}(\sqrt{2})$. Moreover $(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + b) + (c + d)\sqrt{2}$ and $(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2}$. Since the rationals are closed under addition and multiplication, we see that $\mathbb{Q}(\sqrt{2})$ is as well. Finally, if $a + b\sqrt{2} \in \mathbb{Q}(\sqrt{2})$, then its additive inverse $-a - b\sqrt{2}$ is also in $\mathbb{Q}(\sqrt{2})$, as is its multiplicative inverse, $\frac{a}{a^2+2b^2} - \frac{b}{a^2+2b^2}\sqrt{2}$.

(b) Let $\boxplus$ be an order on $\mathbb{Q}(\sqrt{2})$ which gives it the structure of an ordered field. First, we know that there is exactly one way of making $\mathbb{Q}$ into an ordered field, so for numbers of the form $a = a + 0\sqrt{2}$, we must have $a < b \Leftrightarrow a \boxplus b$.

Now we observe that if $0 \boxplus p, q$ are distinct positive numbers and $p^2 \boxplus q^2$, then $p \boxplus q$. For if $q \boxplus p$, then $q^2 \boxplus qp \boxplus p^2$ after applying Proposition 1.18(b) twice.

Suppose that $0 \boxplus \sqrt{2}$. Then for any rational b such that $0 \boxplus b$, we have $0 \boxplus b\sqrt{2}$, and $(b\sqrt{2})^2 = 2b^2$ is rational with $0 \boxplus 2b^2$. Since we know how to order the rationals, and we know how to order any positives if we know how to order their squares, we conclude that for numbers x, x' each of which of the form $a + 0\sqrt{2}$ or $0 + b\sqrt{2}$ for $a, b > 0$, we must have $x \boxplus x' \Leftrightarrow x < x'$. Taking additive inverses implies that the same is true of all elements of the form $a + 0\sqrt{2}$ or $0 + b\sqrt{2}$ such that $a, b \in \mathbb{Q}$.

Now, consider $a+b\sqrt{2}$ and $c+d\sqrt{2}$. We see that $a+b\sqrt{2} \boxplus c+d\sqrt{2} \Leftrightarrow a-c \boxplus d-b\sqrt{2} \Leftrightarrow a-c < d-b\sqrt{2} \Leftrightarrow a+b\sqrt{2} < c+d\sqrt{2}$. So we have ordered all elements of $\mathbb{Q}(\sqrt{2})$, and the order is the order inherited from $\mathbb{R}$.

The second case is $\sqrt{2} \boxplus 0$. We can repeat the argument above using $-\sqrt{2}$ in place of $\sqrt{2}$ to show this uniquely determines a second order on $\mathbb{Q}(\sqrt{2})$.(This corresponds to the fact that 2 has a positive and a negative square root and the construction of this field doesn't notice which one we chose.)

3. (a) Let $A = \{x\}$ and B be closed. Let $S = \{d(x, y) : y \in B\}$ and $\alpha = d(A, B) = \inf S$. For all $n \in \mathbb{N} - \{0\}$, $\alpha + \frac{1}{n}$ is not a lower bound for S, so we can choose y_n such that $d(x, y_n) < \alpha + \frac{1}{n}$. Then $\{y_n\} \subset N_{\alpha+1}(x)$, so $\{y_n\}$ is a bounded sequence in $\mathbb{R}^k$, implying that $\{y_n\}$ has a convergent subsequence $y_{n_k} \to y$. Since B is closed, $y \in B$. Given any $m > 0$, choose K such that $k > K$ implies $d(y_{n_k}, y) < \frac{1}{m}$. Then for $n_k > \max\{n_K, m\}$, we have $d(x, y) \leqslant d(x, y_{n_k}) + d(y_{n_k}, y) < \alpha + \frac{1}{m} + \frac{1}{m} = \alpha + \frac{2}{m}$. As m was arbitrary, $d(x, y) = \alpha$.

(b) Let A be compact and B be closed. Then let $S = \{d(x, y) : x \in A, y \in B\}$ and $\alpha = d(A, B) = \inf S$. For all $n \in \mathbb{N} = \{0\}$, pick $x_n \in A$ and y_n in B such that $d(x_n, y_n) < \alpha + \frac{1}{n}$. As A is compact, some subsequence x_{n_k} converges to a point x in A. Then for any $m \in \mathbb{N} - \{0\}$, there exists K such that $k \geqslant K$ implies that $d(x_{n_k}, x) < \frac{1}{2m}$. For $k \geqslant \max\{K, 2m\}$, we have $d(x, y_{n_k}) < d(x, x_{n_k}) + d(x_{n_k}, y_{n_k}) < \frac{1}{2m} + \alpha + \frac{1}{2m} = \alpha + \frac{1}{2m}$. Since B is closed we are now in a position to repeat the argument from part (a) to conclude there is a point $y \in B$ such that $d(x, y) = \alpha$.

(c) Consider the sets $A = \mathbb{N} - \{0\}$ and $B = \{n + \frac{1}{2n} : n \in \mathbb{N} - \{0\}\}$. These sets are both closed (neither of them have limit points), and $A \cap B = \varnothing$, but $d(A, B) = 0$.

4. (a) No; consider the sequence $s_n = n$. I claim this is Cauchy. For given $\epsilon > 0$, choose N such that $\frac{1}{N} < \epsilon$. Then if $n, m \geqslant N$, we have $d(s_n, s_m) < |\frac{1}{n} - \frac{1}{m}| < \frac{1}{N} < \epsilon$. So this sequence is Cauchy. However, suppose it converges to some $x \in [1, \infty)$. Then let $\epsilon = \frac{1}{k}$ for some k a positive natural. By assumption, there exists N such that $n \geqslant N$ implies $\frac{1}{k} > d(x, s_n) = |\frac{1}{x} - \frac{1}{n}|$. For n large enough, this implies that $\frac{1}{k} > \frac{1}{x} - \frac{1}{n} > 0$, and therefore $\frac{1}{k} \geqslant \frac{1}{x}$. Hence $x \geqslant k$. But k was arbitrary, so no such x exists.

   (b) No; consider $[1, \infty)$ itself. Notice for any $x \in [1, \infty)$, $d(1, x) = |1 - \frac{1}{x}| < 1$, so $[1, \infty) = N_1(x)$ and $[1, \infty)$ is bounded. Morever it is closed as a subset of itself, as all metric spaces are. But it is certainly not compact, since by part (a) it is not complete.

   (Identifying this space with $(0, 1]$ with the standard metric and solving the problem that way is also a fine solution.)

5. (a) Let $(x, y) \in X \times Y$. Then

$$\begin{aligned} N_r((x,y)) &= \{(x', y') : d((x,y),(x',y')) < r\} \\ &= \{(x', y') : \sup\{d_X(x,x'), d_Y(y,y')\} < r\} \\ &= \{(x', y') : d_X(x,x'), d_Y(y,y') < r\} \\ &= \{(x', y') : x' \in N_r(x), y' \in N_r(y)\} \\ &= N_r(x) \times N_r(y). \end{aligned}$$

   (b) Since compactness is not relative, it suffices to prove the statement for $X = K$ and $Y = L$. First, observe that for any $E \subset X$ and $y \in Y$, if (x, y) and (x', y) are points in $E \times \{y\} \subset X \times Y$, then $d((x, y), (x', y)) = d_X(x, x')$. So the inherited metric on $E \times \{y\}$ is exactly the metric on E. Similarly if $F \subset Y$, the inherited metric on $\{x\} \times F$ agrees with the metric on F for all x.

   First suppose $X \times Y$ is compact. Pick some $y \in Y$. Then $X \times \{y\}$ is a copy of X in $X \times Y$. Moreover I claim $X \times \{y\}$ is closed, since if $(x', y') \notin X \times \{y\}$ then $N_{d_Y(y,y')}(x', y')$ contains no point of $X \times \{y\}$. Ergo $X \times \{y\}$ is a closed subset of a compact set, hence compact. We conclude that X is compact. Likewise Y is compact.

   In the other direction, suppose that X is compact and Y is compact. Let $\{U_\alpha\}$ be an open cover of $X \times Y$. Given $y \in Y$, the set $X \times \{y\}$ is compact, so we can pick finitely many $U_1^y, \ldots, U_{n_y}^y$ which cover $X \times \{y\}$. Indeed, for any $x \in X$, there is a neighborhood $V_x = N_{r_x}((x, y)) = N_{r_x}(x) \times N_{r_x}(y)$ contained in U_j for some j. Since $X \times \{y\}$ is compact, we can choose finitely many $V_{x_1}, \ldots, V_{x_m}$ which cover $X \times \{y\}$. Then if $r_y = \min\{r_{x_1}, \ldots, r_{x_m}\}$, we see that $X \times N_{r_y}(y) \subset \cup_{i=1}^m V_{x_m} \subset \cup_{i=1}^{n_y} U_i^y$. But Y is compact, so we can choose finitely many $y_1, \ldots, y_k$ such that the neighborhoods $N_{r_y}(y_j)$ cover Y. Then $X \times Y$ is covered by the finite subcover $\{U_\ell^{y_j} : 1 \leqslant j \leqslant k, 1 \leqslant \ell \leqslant n_{y_j}\}$ of $\{U_\alpha\}$. Since $\{U_\alpha\}$ was arbitrary, we conclude that $X \times Y$ is compact.

   [There is also a faster proof of this using subsequences. Exercise!]

6. (a) Let $x_n = n^{\frac{1}{n}} - 1$. By Theorem 3.3(a), it suffices to check that $x_n \to 0$, since then

$n^{\frac{1}{n}} = x_n + 1 \to 1$. Now observe that $x_n \geqslant 0$, and furthermore

$$\begin{aligned} n &= (1+x_n)^n \\ &= \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} x_n^k \\ &> \frac{n!}{(n-2)!2!} x_n^2 \\ &= \frac{n(n-1)}{2} x_n^2 \end{aligned}$$

We conclude that $0 \leqslant x_n \leqslant \sqrt{2n-1}$. Now for $\epsilon > 0$, choose N such that $N > 1 - \frac{2}{\epsilon^2}$, so that $\sqrt{2N-1} < \epsilon$. We see that $n \geqslant N$ implies that $|x_n - 0| = |x_n| = x_n < \sqrt{2n-1} < \sqrt{2N-1} < \epsilon$. So $x_n \to 0$ as desired.

(b) Let $s_n \to 0$ and (t_n) be bounded, say $|t_n| < M$. Then for any $\epsilon > 0$, choose N such that $n \geqslant N$ implies that $|s_n| = |s_n - 0| < \frac{\epsilon}{M}$. We see that if $n \geqslant N$, we have $|s_n t_n - 0| = |s_n t_n| < |s_n| M < \frac{\epsilon}{M} \cdot M = \epsilon$. As ϵ was arbitary, we have $s_n t_n \to 0$.

7. (a) Consider the set $S = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N} - \{0\}\} \cup \{\frac{1}{m} + \frac{1}{n} : n, m \in \mathbb{N} - \{0\}, n > m\}$. Clearly 0 and $\frac{1}{m}$ for all $m \in \mathbb{N} - \{0\}$ are limit point of S. We must check it has no other limit points. Certainly if $x \in \mathbb{R}$ has $x < 0$ or $x > 2$, then x has some neighborhood containing no point of S and is therefore not a limit point of S. Furthermore, if $x \in [0,2]$ is not 0 or $\frac{1}{m}$ for some m, then choose m such that $\frac{1}{m+1} < x < \frac{1}{m}$. Then pick $\epsilon = \frac{1}{2}\min\{x - \frac{1}{m+1}, \frac{1}{m} - x\}$. Then the neighborhood $N_\epsilon(x)$ about x has finite intersection with S, consisting only of pairs $\frac{1}{m} + \frac{1}{n}$ such that $\frac{1}{n} > \frac{1}{2}(x - \frac{1}{m+1})$. Hence x is not a limit point of S. So S is closed; since it is also bounded and is a subset of $\mathbb{R}$, S is compact.

   (b) Consider the set of points in $[0,1]$ with only 4's and 7's in their decimal expansions from class. We translate this set by adding $.10100100010000100000\ldots$ to all of its elements. The result is a set of elements whose decimal expansions never become repeating. This set is still perfect, since we did not change its topology via the translation.

8. (a) Let $x = (x_1, x_2, \ldots)$ be an element of S. Let $r > 0$, and let k be the largest integer such that $\frac{1}{2^k} < r$. Then $N_r(x) = \{(y_1, y_2, \ldots) : y_i = x_i \forall i < k\}$. Moreover, if y is any element of $N_r(x)$, we see that $N_r(x) = N_r(y)$. Now suppose the intersection of two neighborhoods $N_r(x)$ and $N_s(z)$ is nonempty. Wlog let $r > s$. Choose some y in the intersection. Then $N_r(x) = N_r(y)$ and $N_s(z) = N_s(y)$, and therefore $N_s(z) = N_s(y) \subseteq N_r(y) = N_r(x)$.

   (b) Let $\{x^i\} = \{(x_1^i, x_2^i, \ldots)\}$ be a sequence which converges to some x in S. Then for any $k > 0$ an integer, there exists N such that $n \geqslant N$ implies that $d(x^n, x) < \frac{1}{2^k}$, implying that $x_j^n = x_j$ for all $1 \leqslant j \leqslant k$. Since n was arbitrary, any x^n and x^m for which $n, m \geqslant N$ have the same first k entries. So the convergent sequences in S are exactly those sequences such that for all k, there exists N such $n, m \geqslant N$ implies that $x_j^n = x_j^m$ for all $1 \leqslant j \leqslant k$.