MTH 327H: Exam 1

Due Date: Friday, October 19, at 10:20 a.m.

Instructions: Please submit complete solutions to **exactly five** of the questions below. If you submit more than five solutions, the first five on the paper you hand in will be graded. Each problem is worth ten points. Problems are **not in order of difficulty.** Do not use any resources other than the textbook and your notes and homework to work on the exam. Do not discuss the problems with anyone (other than me). If you spot anything you are concerned is a potential typo, please email me for confirmation or correction immediately.

1. Consider the following metrics on $\mathbb{R}^2 = \{(x_1, x_2) : x_i \in \mathbb{R}\}$.

- The standard metric $d_1(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.
- The taxicab metric $d_2(\vec{x}, \vec{y}) = |x_1 - y_1| + |x_2 - y_2|$.
- The ℓ^∞ metric $d_3(\vec{x}, \vec{y}) = \sup\{|x_1 - y_1|, |x_2 - y_2|\}$.

(a) Two metrics d and d' on a set X are said to be *strongly equivalent* if there exist $0 < \alpha < \beta$ such that for any $p, q \in X$ we have $\alpha d(p, q) \leqslant d'(p, q) \leqslant \beta d(p, q)$. Prove that the three metrics above are pairwise strongly equivalent.

(b) Prove that if a set U is open in $(\mathbb{R}^2, d_i)$ for any of $i = 1, 2, 3$, then U is also open in $(\mathbb{R}^2, d_j)$ with d_j either of the other two metrics. We say that these three metrics *induce the same topology* on $\mathbb{R}^2$. [An analogous statement is true in $\mathbb{R}^k$ by essentially the same argument.]

2. Consider the set $\mathbb{Q}(\sqrt{2}) = \{r + s\sqrt{2} : r, s \in \mathbb{Q}\} \subset \mathbb{R}$.

(a) Prove that $\mathbb{Q}(\sqrt{2})$ is a field with the operations of addition and multiplication inherited from the real numbers. [Hint: Your solution does not need to have eleven parts.]

(b) Prove that there are exactly two ways to equip $\mathbb{Q}(\sqrt{2})$ with the structure of an ordered field.

3. Let A and B be subsets of $\mathbb{R}^k$, and define

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

(a) Let $A = \{x\}$ and B is closed. Prove that $d(A, B) = d(x, y)$ for some $y \in B$.

(b) Suppose that A is compact and B is closed. Prove that $d(A, B) = d(x, y)$ for some $x \in A, y \in B$.

(c) Exhibit closed sets A and B in $\mathbb{R}^k$ for some k such that the condition in part (*b*) fails to hold.

[Note: You can do this with just material from Chapter 2, but you may find it useful to use Theorem 3.6 in Rudin to streamline parts (a) and (b).]

4. Consider the metric space X whose points are the elements of the subset $[1, \infty)$ of the reals, but with the metric $d(x, y) = |\frac{1}{x} - \frac{1}{y}|$.

(a) Is X a complete metric space?

(b) Is it true that if S is closed and bounded in X, then S is compact?

5. Let (X, d_X) and (Y, d_Y) be metric spaces. Consider the metric on $X \times Y$ given by

$$d((x_1, y_1), (x_2, y_2)) = \sup\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

(a) Confirm that $N_r((x, y)) = N_r(x) \times N_r(y)$.

(b) Let $K \subset X$ and $L \subset Y$. Prove that $K \times L \subset X \times Y$ is compact if and only if K and L are both compact. [Hint: Given an open cover of $K \times L$, it may be useful to start by considering finite subcovers of $\{x\} \times L$ for $x \in K$.]

6. (a) Using the definition of the limit of a sequence, prove that

$$\lim_{n \to \infty} n^{\frac{1}{n}} = 1.$$

Rudin outlines one proof of this fact. You can use his proof, but you need to fill in the details. Note that the Binomial Theorem says that $(a + b)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} a^{n-k} b^k$.

(b) Let (s_n) be a sequence of complex numbers such that $s_n \to 0$, and let (t_n) be a bounded sequence of complex numbers; that is, there exists M such that $|t_n| < M$ for all n. Prove that $s_n t_n \to 0$.

7. (a) Construct a compact set of real numbers whose limit points form a countable set.

(b) Construct a non-empty perfect subset of $\mathbb{R}$ which contains no rational number. [There are several ways to do this problem, but one way involves using the fact that the irrationals are exactly the numbers with decimal expansions that never become repeating.]

8. Consider the set S of sequences in $\{0, 1\}$, with the metric

$$d((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = 2^{-k}$$

where k is the smallest integer for which $x_k \neq y_k$. [You do not need to check this is a metric.]

(a) Prove that if U_i and U_j are two neighborhoods in S, then either U_i and U_j have empty intersection or one of U_i or U_j is a subset of the other.

(b) Describe the set of convergent sequences in S.