MTH 327H: Challenge Problems I

1. Problems 7 and 20 in Rudin Chapter 1 are recommended if you haven't seen them previously (they won't be on the regular homework).

2. Prove that the nth root of any positive integer p is either an integer or irrational (that is, if is never in $\mathbb{Q} - \mathbb{Z}$).

3. Suppose that W and M are finite sets, and that to each $m \in M$ is associated a subset W_m of W. Show that the following are equivalent:

   - There exists an injective map $f\colon M \to W$ such that $f(m) \in W_m$ for every $m \in M$.
   - For every non-empty subset $X \subset M$, $\mathrm{Card}(\bigcup_{m \in X} W_m) \geqslant \mathrm{Card}(X)$.

4. Prove the Schroder-Bernstein Lemma: If X and Y are sets such that there are injective maps $f\colon X \to Y$ and $g\colon Y \to X$, then there is a bijection from X to Y.

5. Let S be a collection of elements of $P(\{1, \cdots, n\})$ such that for any $X, Y \in S$ we have $X \nsubseteq Y$.

   (a) Show that $\Sigma_{X \in S} \mathrm{Card}(X)!(n - \mathrm{Card}(X))! \leqslant n!$

   (b) Prove Sperner's Lemma, that $\mathrm{Card}(S) \leqslant \binom{n}{\lfloor n/2 \rfloor}$

6. Let $\mathbb{R}(x)$ be the set of rational functions with coefficients in $\mathbb{R}$. Show that $\mathbb{R}(x)$ with the natural notions of addition and multiplication can be given the structure of an ordered field by letting the set of positive elements be those functions which can be written $\frac{P(x)}{Q(x)}$ such that the leading coefficients of both P and Q are positive. Prove that $\mathbb{R}(x)$ is non-Archimedean (hence not complete).

The following exercises use more background than we've built up in class so far.

1. Let $\mathcal{C}(\mathbb{R}, \mathbb{R})$ be the set of continuous functions from $\mathbb{R} \to \mathbb{R}$. Show this set has the same cardinality as $\mathbb{R}$.

2. Prove that the Cantor set C is the "universal compact metric space"; that is, prove that for any compact nonempty metric space M, there is a continuous surjection from C to M.

3. Give an example of a subset E of $\mathbb{R}$ such that E, $\overline{E}$, E°, $(\overline{E})^\circ$, $\overline{(E^\circ)}$, $\overline{((\overline{E})^\circ)}$, and $(\overline{(E^\circ)})^\circ$ are all different.

4. Prove that any subfield of $\mathbb{C}$ which is not contained in $\mathbb{R}$ is a dense subset of $\mathbb{C}$.

5. A set X in a metric space is called thin if it can be written as the countable union of closed sets each having empty interior. A set Y is called negligible or measure zero if for any $\epsilon > 0$ we can find a covering of Y by countably many open balls, the sum of whose volumes is less than ϵ. (The volume of an open ball in $\mathbb{R}^k$ is what you think it is, in particular, the volume of (a, b) in $\mathbb{R}$ is $b - a$.) Prove that the real numbers can be decomposed as $X \cup Y$ where X is thin and Y is negligible.

6. (I have no idea whether the people reading this sheet have seen the p-adics. If you would like a simple construction to go here, send me an email.)

7. Stay tuned...