MTH 320, Section 003
Analysis

Sample Midterm 2

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books or notes. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1.

(a) [5pts.] Let $f: \mathbb{R} \to \mathbb{R}$. What does it mean to say that f is continuous at x_0?

> **Solution:** We say that f is continuous at x_0 if for every sequence of real numbers (x_n) such that $x_n \to x_0$, we have $f(x_n) \to f(x_0)$.

(b) [5pts.] A set S is said to be *dense* in $\mathbb{R}$ if every open interval contains a point in S. (For example, both the rationals and the irrationals are dense in $\mathbb{R}$.) Suppose S is dense in $\mathbb{R}$, $f, g: \mathbb{R} \to \mathbb{R}$ are continuous on $\mathbb{R}$, and $f(s) = g(s)$ for every $s \in S$. Prove that $f(x) = g(x)$ for every $x \in \mathbb{R}$.

> **Solution:** Suppose that $x \notin S$. Then for every $n \in \mathbb{N}$, we can find an $s_n \in S$ lying in $(x - \frac{1}{n}, x + \frac{1}{n})$. Then by construction $s_n \to x$. Ergo $f(x) = \lim f(s_n) = \lim g(s_n) = g(x)$, since f and g are equal on S. Ergo f and g are equal on all of $\mathbb{R}$.

Problem 2.

For each of the following, either give an example of a power series with the given properties, or prove that one cannot exist.

(a) [3pts.] A power series with interval of convergence $(0, 2]$.

> **Solution:** Note that the center of this power series, if it exists, is 1, and the radius of convergence is 1. Consider the power series $\sum_{n-1}^{\infty} \frac{(-1)^n}{n}(x-1)^n$. Then we see that $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{n}{n+1} = 1$, so the radius of convergence of this power series is $R = 1$. Hence the power series converges absolutely on $(0, 2)$, and diverges on $|x - 1| > 1$. At $x = 0$ we have the power series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(-1)^n = \sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. At $x = 2$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which converges. Ergo the interval of convergence of this power series is $(0, 2]$.

(b) [4pts.] A power series which converges uniformly on its interval of convergence.

> **Solution:** Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}x^n$. From lecture, the interval of convergence of this power series is $[-1, 1]$. In particular, we observe that for any $x\ in [-1, 1]$, $|\frac{1}{n^2}x^n| \leq |\frac{1}{n^2}|$, and $\sum \frac{1}{n^2}$ converges absolutely. Ergo by the Weierstrass M-test, the power series converges uniformly on $[-1, 1]$.

(c) [3pts.] A power series with interval of convergence $[2, 3]$.

> **Solution:** Note that the center of this power series, if it exists, is $\frac{5}{2}$ and the radius of convergence is $\frac{1}{2}$. Consider $\sum_{n=0}^{\infty} \frac{2^n}{n^2}(x - \frac{5}{2})^n$. We see that $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{2^{n+1}(n)^2}{2^n(n+1)^2} = 2$, so the radius of convergence of this power series is $R = \frac{1}{2}$.

Hence the power series converges absolutely on $(2,3)$, and diverges on $|x-\frac{5}{2}|>\frac{1}{2}$. At the endpoint $x=2$ we have $\sum_{n=1}^{\infty}\frac{2^n}{n^2}(-\frac{1}{2})^n=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$, which converges. At the endpoint $x=3$ we have $\sum_{n=1}^{\infty}\frac{2^n}{n^2}(\frac{1}{2})^n=\sum_{n=1}^{\infty}\frac{1}{n^2}$, which also converges. Ergo the interval of convergence of the power series is $[1,2]$.

Editorial Note: The question roughly corresponding to this one on the actual midterm is somewhat shorter.

Problem 3.

(a) [5pts.] Let $f:\mathbb{R}\to\mathbb{R}$. What does it mean to say that f is differentiable at a?

Solution: We say that f is differentiable at a if the limit

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

exists, and we call this limit $f'(x)$.

(b) [5pts.] Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be two real-valued functions such that $g(a)\neq 0$, and both f and g are differentiable at a. Prove that $\frac{f}{g}$ is differentiable at a.

Solution: First, since g is differentiable at a, g must be continuous at a, so since $g(a)\neq 0$ there is a small neighborhood around a on which $g(x)\neq 0$ and it makes sense to consider $\frac{f}{g}(x)$. Now observe that

$$\begin{aligned}\frac{\frac{f}{g}(x)-\frac{f}{g}(a)}{x-a}&=\frac{1}{g(x)g(a)}\left(\frac{f(x)g(a)-g(x)f(a)}{x-a}\right)\\&=\frac{1}{g(x)g(a)}\left(\frac{f(x)g(a)+f(a)g(a)-g(a)f(a)-g(x)f(a)}{x-a}\right)\\&=\frac{1}{g(x)g(a)}\left(\frac{f(x)-f(a)}{x-a}\cdot g(a)-f(a)\cdot\frac{g(x)-g(a)}{x-a}\right)\end{aligned}$$

Because $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=f'(a)$, $\lim_{x\to a}\frac{g(x)-g(a)}{x-a}=g'(a)$, and $\lim_{x\to a}g(x)=g(a)$, it follows from the limit laws that the limit of the last expression as $x\to a$ is $\frac{f'(a)g(a)-g'(a)f(a)}{g(a)^2}$. In particular this limit exists and is a real number, so $\frac{f}{g}$ is differentiable at a.

Problem 4.

Let (f_n) be a sequence of real-valued functions.

(a) [5pts.] What does it mean for (f_n) to converge uniformly to f on a domain $S\subset\mathbb{R}$?

Solution: We say that $f_n \to f$ uniformly on S if for every $\epsilon > 0$, there is a N such that for any $x \in S$ and $n > N$, we have $|f_n(x) - f(x)| < \epsilon$.

(b) [5pts.] Suppose that (f_n) is uniformly continuous on S, and $f_n \to f$ uniformly on S. Prove that f is uniformly continuous on S.

Solution: Let $\epsilon > 0$. There exists N such that for $n \geq N$ and $x \in S$, $|f_n(x) - f(x)| < \frac{\epsilon}{3}$. In particular, for every $x \in S$, $|f_N(x) - f(x)| < \frac{\epsilon}{3}$. Moreover, there exists δ such that if $x, y \in S$ and $|x - y| < \delta$, we have that $|f_N(x) - f_N(y)| < \frac{\epsilon}{3}$. Therefore if $x, y \in S$ and $|x - y| < \delta$, we have

$$\begin{aligned}|f(x) - f(y)| &\leq |f_x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon.\end{aligned}$$

Problem 5.

Let (a_n) be a sequence of positive numbers such that $\lim a_n = 0$.

(a) [5pts.] Give an example to show that $\sum a_n$ need not converge.

Solution: We know that $\frac{1}{n} \to 0$ but $\sum_{n=0}^{\infty} \frac{1}{n} = \infty$.

(b) [5pts.] Prove that there exists a subsequence (a_{n_k}) of (a_n) such that $\sum_{k=1}^{\infty} a_{n_k}$ converges.

Solution: Because $\lim a_n = 0$, there exists N_1 such that for $n > N_1$, $a_n < 1$. Pick some $n_1 > N_1$, so that $a_{n_1} < 1$. Now, there is some N_2 such that for $n > N_2$, $a_n < \frac{1}{4}$. Pick n_2 such that $n_2 > \max\{n_1, N_1\}$, so that $a_{n_2} < \frac{1}{4}$. Iterate this process, choosing n_k such that $n_k > n_{k-1}$ and $a_{n_k} < \frac{1}{k^2}$. Then we see that $\sum_{k=1} a_{n_k}$ converges by comparison to $\sum \frac{1}{n^2}$.

This page is for scratch work. Feel free to tear it off. Do not write anything you want graded on this page unless you indicate *very clearly* that this is the case on the page of the corresponding problem.