MTH 320, Section 003
Analysis

Sample Midterm 1

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books or notes. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1.

(a) [5pts.] Let F be a field, and $\leq$ an order relation. List the axioms that $\leq$ must satisfy.

Solution:

1. (Totality) If a, b in F, either $a \leq b$ or $b \leq a$.
2. (Antisymmetry) If $a \leq b$ and $b \leq a$, then $a = b$.
3. (Transitivity) If $a \leq b$ and $b \leq c$, then $a \leq c$.
4. (Additive Invariance) If $a \leq b$ and $c \in F$, then $a + c \leq b + c$.
5. (Multiplicative Invariance) If $a \leq b$ and $c \geq 0$, then $ac \leq bc$.

(b) [5pts.] Let $F = \{0, 1\}$. F can be given the structure of a field with addition and multiplication

$$\begin{aligned} 0 + 0 &= 0 & \qquad 0 \times 0 &= 0 \\ 1 + 0 &= 1 & \qquad 1 \times 0 &= 0 \\ 0 + 1 &= 1 & \qquad 0 \times 1 &= 0 \\ 1 + 1 &= 0 & \qquad 1 \times 1 &= 1 \end{aligned}$$

Show that F cannot be given the structure of an ordered field.

Solution: Recall that in an ordered field, if $a \leq c$, then $a+b \leq a+c$. Moreover, $0 \leq 1$, so $0 + 1 \leq 1 + 1$, and therefore $1 \leq 0$. But by the second order axiom, if $1 \leq 0$ and $0 \leq 1$, then $1 = 0$, which is impossible.

Problem 2.

Let $S \subset \mathbb{R}$ be a nonempty bounded subset of $\mathbb{R}$.

(a) [5pts.] Define the supremum and infimum of S.

Solution: We say b is the supremum of S if b is an upper bound for S and, if M is any upper bound for S, then $b \leq M$. Similarly, we say c is the infimum of S if c is a lower bound for S and if m is any lower bound for S, then $m \leq c$.

(b) [5pts.] Let S and T be two subsets of $\mathbb{R}$ which are bounded above. Let $S \cup T$ be their union, i.e. $x \in S \cup T$ if and only if $x \in S$ or $x \in T$. Show that $\sup(S \cup T) = \max\{\sup S, \sup T\}$.

Solution: Let $x \in S \cup T$. Then either $x \in S$ or $x \in T$ (or both). If $x \in S$, $x \leq \sup S$, and if $x \in T$, $x \leq \sup T$, so $x \leq \max\{\sup S, \sup T\}$, and therefore $\max\{\sup S, \sup T\}$ is an upper bound for $S \cup T$. Now let M be any other upper bound for $S \cup T$. Then for any $s \in S$, since $s \in S \cup T$, $s \leq M$, so M is an upper bound for S, and therefore $\sup S \leq M$. Similarly, $\sup T \leq M$. Ergo $\max\{\sup S, \sup T\} \leq M$. So $\max\{\sup S, \sup T\}$ is the least upper bound of $S \cup T$, as claimed.

Problem 3.

(a) [5pts.] Define a Cauchy sequence.

Solution: We say a sequence (s_n) is Cauchy if, for every $\epsilon > 0$, there exists N such that if $n, m > N$, then $|s_n - s_m| < \epsilon$.

(b) [5pts.] Prove that if (s_n) and (t_n) are Cauchy, then their product $(s_n t_n)$ is also a Cauchy sequence. (Hint: This is extremely similar to the corresponding proof for convergent sequences.)

Solution: Recall that Cauchy sequences are bounded, and choose M_1 such that $|s_n| \leq M_1$ for all n and M_2 such that $|t_n| \leq M_2$ for all n. Now let $\epsilon > 0$. Pick N_1 so that $n, m > N_1$ implies that $|s_n - s_m| < \frac{\epsilon}{2M_2}$ and N_2 so that $n, m > N$ implies that $|t_n - t_m| < \frac{\epsilon}{2M_1}$. Then when $n, m > \max\{N_1, N_2\}$, we see that

$$\begin{aligned}
|s_n t_n - s_m t_m| &= |(s_n t_n - s_m t_n) + (s_m t_n - s_m t_m)| \\
&\leq |s_n t_n - s_m t_n| + |s_m t_n - s_m t_m)| \\
&= |t_n||s_n - s_m| + |s_m||t_n - t_m| \\
&< M_2 \left(\frac{\epsilon}{2M_2}\right) + M_1 \left(\frac{\epsilon}{2M_1}\right) \\
&= \epsilon
\end{aligned}$$

We conclude that $(s_n t_n)$ is Cauchy.

Problem 4.

Let (s_n) be a sequence of real numbers.

(a) [5pts.] Suppose that (s_n) and (t_n) are bounded sequences of nonnegative numbers. Prove that $\limsup(s_n t_n) \leq (\limsup s_n)(\limsup t_n)$.

Solution: It suffices to prove that $\sup\{s_n t_n : n > N\} \leq \sup\{s_n : n > N\} \cdot \sup\{t_n : n > N\}$ for all N; then taking the limit as $N \to \infty$ proves the statement above for the limit suprema. Let $a = \sup\{s_n : n > N\}$ and $b = \sup\{t_n : n > N\}$.

Then for $n > N$, we have $s_n \le a$ and $t_n \le b$, so $s_n t_n \le ab$, implying that ab is an upper bound for $\{s_n t_n : n > N\}$. Therefore $\sup\{s_n t_n : n > N\} \le ab = \sup\{s_n : n > N\} \cdot \sup\{t_n : n > N\}$, as desired.

(b) [5pts.] Give an example to show that the inequality in part (a) need not be an equality.

Solution: Consider the sequences $(s_n) = (1, 0, 1, 0, 1, 0, \cdots)$ and $(t_n) = (0, 1, 0, 1, \cdots)$. We have that $\limsup s_n = \limsup t_n = 1$, but $\limsup(s_n t_n) = 0$.

Problem 5.

Let s_n be a sequence defined recursively by $s_1 = 10$ and $s_n = \frac{1}{4}(s_{n-1} + 6)$.

(a) [5pts.] Show that (s_n) is decreasing and satisfies $s_n > 2$ for all n.

Solution: We proceed by induction. For the first assertion, the base case is $s_1 = 10 > 4 = s_2$. Inductively, assume $s_{n-1} > s_n$. Then we would like to show that $s_{n+1} < s_n$, or equivalently that $s_{n+1} - s_n < 0$. We compute $s_{n+1} - s_n = \frac{1}{4}(s_n + 6) - \frac{1}{4}(s_{n-1} + 6) = \frac{1}{4}(s_n - s_{n-1}) < 0$ by assumption. So we are done.

For the second claim, the base case is $s_1 = 10 > 2$. For the inductive step, suppose $s_n > 2$. Then $s_{n+1} = \frac{1}{4}(s_n + 6) > \frac{1}{4}(2 + 6) = 2$. So we are done.

(b) [5pts.] Does (s_n) converge? If so, what is the limit? Justify your answer carefully.

Solution: Yes, (s_n) is bounded monotone, so s_n converges to some limit s. In particular, we may use the limit laws to take the limit of both sides of $s_n = \frac{1}{4}(s_{n-1} + 6)$, we see that $s = \frac{1}{4}(s + 6)$, and solving gives $s = 2$.

This page is for scratch work. Feel free to tear it off. Do not write anything you want graded on this page unless you indicate *very clearly* that this is the case on the page of the corresponding problem.