MTH 320, Section 003
Analysis

Sample Final

Instructions: You have two hours to complete the exam. There are eight problems, worth a total of eighty points. You may not use any books or notes. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: ________________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

Problem 1.

(a) [5pts.] State the Mean Value Theorem.

> **Solution:** Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). Then there is some $x \in (a, b)$ such that
> $$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

(b) [5pts.] Suppose that $f(x)$ is differentiable on $\mathbb{R}$ and $f'(x) > 0$ on $\mathbb{R}$. Prove that f is strictly increasing.

> **Solution:** Let $x < y$. We observe that by MVT, there is some $z \in (x, y)$ such that $f'(z)(y-x) = f(y)-f(x)$. But since $f'(z) > 0$, we see that $f(y)-f(x) > 0$, or $f(y) > f(x)$. Ergo f is strictly increasing.

Problem 2.

Let $f\colon \mathbb{R} \to \mathbb{R}$.

(a) [5pts.] What does it mean for f to be differentiable at a?

> **Solution:** We say that f is differentiable at a if the limit
> $$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
> exists and is a real number, and we call this limit $f'(x)$.

(b) [5pts.] Let $f(x) = x \sin(\frac{1}{x})$ when $x \neq 0$ and $f(0) = 0$. Is f differentiable at $x = 0$? Justify your answer.

> **Solution:** We observe that $\frac{f(x)-f(0)}{x} = \frac{x \sin(\frac{1}{x})}{x} = \sin(\frac{1}{x})$. The limit $\lim_{x \to 0} \sin(\frac{1}{x})$ does not exist, so f is not differentiable at $x = 0$.

Problem 3.

(a) [5pts.] Prove that the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges on $[-1, 1)$.

> **Solution:** The power series is centered around 0, and an application of the ratio test shows that the radius of convergence is 1. As for the endpoints, at $x = -1$ we have the alternating harmonic series, which converges, and at $x = 1$ we have the harmonic series, which diverges.

(b) [5pts.] What function does the power series above represent on $(-1, 1)$? Justify your answer.

Solution: Recall that power series can be differentiated term by term. Indeed, the derivative of $f(x)$ is $\sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^n$, which converges to $\frac{1}{1-x}$ on $(-1, 1)$. Antidifferentiating and noting that $f(0) = 0$, we see that $f(x) = -\ln(1-x)$ on $(-1, 1)$.

Problem 4.

(a) [5pts.] Define $\liminf s_n$.

Solution: We say $\liminf s_n = \lim_{N\to\infty} \inf\{s_n : n > N\}$.

(b) [5pts.] Prove that if $\liminf s_n = \limsup s_n = s$ then (s_n) converges to s.

Solution: Since $s = \limsup s_n$, there exists a positive number N_1 such that $|s - \sup\{s_n : n > N_1\}| < \epsilon$. In particular, $\sup\{s_n : n > N_1\} < s + \epsilon$, so for $n > N_1$, $s_n < s + \epsilon$. Similarly, since $\limsup s_n = s$, there exists N_2 such that $n > N_2$ implies that $s - \epsilon < s_n$. Therefore if $n \geq \max\{N_1, N_2\}$, $s - \epsilon < s_n < s_+\epsilon$, so $|s - s_n| < \epsilon$. Ergo $s_n \to s$.

Problem 5.

(a) [5pts.] State the Weierstrass M-test.

Solution: Let $\sum_{k=0}^{\infty} g_k(x)$ be a series of functions on a domain S such that $|g_k(x)| \leq M_k$ on S for some constant M_k, and $\sum_{k=0}^{\infty} M_k$ converges. Then $\sum_{k=0}^{\infty} g_k(x)$ converges uniformly.

(b) [5pts.] Prove that the series of functions $\sum_{n=1}^{\infty} \frac{1}{n^3} \sin(nx)$ converges to a continuous function on all of $\mathbb{R}$, being careful to justify all of your steps.

Solution: Observe that $|\frac{1}{n^3} \sin(nx)| \leq |\frac{1}{n^3}| = M_n$ on $\mathbb{R}$, and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges. Ergo $\sum_{n=1}^{\infty} \frac{1}{n^3} \sin(nx)$ converges uniformly. But each $\frac{1}{n^3} \sin(nx)$ is continuous, so since the series converges uniformly, the limit function is also continuous.

Problem 6.

(a) [5pts.] State the Bolzano-Weierstrass Theorem.

Solution: Every bounded sequence has a convergent subsequence.

(b) [5pts.] Let $S \subset \mathbb{R}$. Suppose that every sequence (s_n) in S has a subsequence converging to an element of S. Prove that S must be closed and bounded. [Hint: Give two arguments to establish that if S is not closed or not bounded, you can produce a sequence with no such subsequence.]

Solution: Suppose S is not bounded. If in particular S is not bounded above, for any $n \in \mathbb{N}$, we can choose an $x_n \in S$ such that $x_n > n$. Then the sequence (x_n) diverges to infinity, as does any subsequence of (x_n). Ergo (x_n) has no convergent subsequence. If S is not bounded below, the same argument holds for a sequence diverging to $-\infty$.

Now suppose that S is not closed. That means there is at least one sequence (x_n) such that $x_n \in S$ but $x_n \to x \notin S$. Then every subsequence of (x_n) also converges to x, and in particular does not converge to any element of S.

Problem 7.

(a) [5pts.] State L'Hospital's Rule.

Solution: Let $s \in \{a, a^\pm, \pm\infty\}$, and let $\lim_{x\to s} \frac{f'(x)}{g'(x)} = L$. Then if either $\lim_{x\to s} f(x) = 0 = \lim_{x\to s} g(x)$ or $\lim_{x\to s} |g(x)| = \infty$, the limit $\lim_{x\to s} \frac{f(x)}{g(x)}$ exists and is also L.

(b) [5pts.] Find the following limits.

- $\lim_{y\to\infty}(1+\frac{2}{y})^y$
- $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

Solution:

We rewrite the first limit as $\lim_{y\to\infty} e^{y\ln(1+\frac{2}{y})}$. Furthermore, we observe that

$$\begin{aligned}
\lim_{y\to\infty} y \ln\left(1+\frac{2}{y}\right) &= \lim_{y\to\infty} \frac{\ln(1+\frac{2}{y})}{\frac{1}{y}} \\
&= \lim_{y\to\infty} \frac{\frac{-2}{y^2(1+\frac{2}{y})}}{-\frac{1}{y^2}} \\
&= \lim_{y\to\infty} \frac{2}{1+\frac{2}{y}} \\
&= 2
\end{aligned}$$

Here the second equality is by L'Hospital's Rule and relies on the final limit existing. So we see that the overall limit $\lim_{y\to\infty}(1+\frac{2}{y})^{\frac{1}{y}} = e^2$.

For the second limit, we multiply by $\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$ to obtain

$$\lim_{x\to 0} \frac{(1+x)-(1-x)}{x(\sqrt{1+x}+\sqrt{1-x})} = \lim_{x\to 0} \frac{2}{(\sqrt{1+x}+\sqrt{1-x})} = 1.$$

Problem 8.

(a) [5pts.] Define the expression $\lim_{x\to a^+} f(x) = L$.

Solution: We say that $\lim_{x\to a^+} f(x) = L$ if there is an interval (a, b) such that if (x_n) is a sequence in (a, b) converging to a, then $(f(x_n))$ converges to L.

(b) [5pts.] Prove that if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$, then $\lim_{x\to a} f(x) = L$.

Solution: We know there is some interval (c, a) such that any sequence (x_n) in (c, a) converging to a has $f(x_n) \to L$, and also some interval (a, b) such that any sequence (x_n) in (a, b) converging to a has $f(x_n) \to L$. Let (t_n) be any sequence in $(c, a) \cup (a, b)$, and divide t_n into two subsequences (t_{n_k}) consisting of those t_n such that $t_{n_k} < a$ and (t_{m_ℓ}) consisting of those t_{m_ℓ} such that $t_{m_\ell} > a$. Then for $\epsilon > 0$, because the left-hand limit is L, there is some N_1 such that $n_k > N_1$ implies $|f(t_{n_k}) - L| < \epsilon$, and because the right-hand limit is L, there is some N_2 such that $m_\ell > N_2$ implies $|f(t_{m_\ell}) - L| < \epsilon$. So for $n > \max\{N_1, N_2\}$, we must have $|f(t_n) - L| < \epsilon$. Since ϵ was arbitrary, $f(t_n) \to L$; since (t_n) was arbitrary, $\lim_{x\to a} f(x) = L$.

This page is for scratch work. Feel free to tear it off. Do not write anything you want graded on this page unless you indicate *very clearly* that this is the case on the page of the corresponding problem.