**MTH 320, Section 003**
**Analysis**

**Quiz 1**

**Instructions:** You have 20 minutes to complete the quiz. There are two problems, worth a total of ten points. You may not use any books or notes. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: ______________________________

| Question | Points | Score |
|---|---|---|
| 1 | 5 | |
| 2 | 5 | |
| Total: | 10 | |

**Problem 1.**

(a) [2pts.] State the Rational Zeroes Theorem.

> **Solution:** Suppose that $r \in \mathbb{Q}$ is a zero of the polynomial $f(x) = c_n x^n + \cdots + c_1 x + c_0$, where $c_i \in \mathbb{Z}$ and $c_n, c_0 \neq 0$. If $r = \frac{a}{b}$ with $(a, b) = 1$, then $a|c_0$ and $b|c_n$.

(b) [3pts.] Prove that $(1 + \sqrt{2})^{\frac{1}{5}}$ is not a rational number.

> **Solution:** Let $a = (1 + \sqrt{2})^{\frac{1}{5}}$. Then $a^5 = 1 + \sqrt{2}$, so $(a^5 - 1)^2 = 2$, implying that $a^{10} - 2a^5 + 1 - 2 = 0$. So $a$ is a zero of the polynomial $f(x) = x^{10} - 2a^5 - 1 = 0$. By the Rational Zeroes Theorem, the only possible rational zeroes of this polynomial of the form $r = \frac{a}{b}$ such that $a|(-1)$ and $b|1$. This implies that the only possible rational zeroes of $f(x)$ are $\pm 1$. But neither of these solves the equation. Therefore $a$ is not a rational number. (Arguments from the closedness of the rationals and the fact that $\sqrt{2}$ is known to be irrational will also be accepted.)

**Problem 2.**

Let $S \subset \mathbb{R}$ be a nonempty subset of $\mathbb{R}$.

(a) [2pts.] Define the supremum of $S$.

> **Solution:** Let $b$ be an upper bound for $S$ such that, given $M$ any upper bound for $S$, $b \leq M$. Then $b$ is said to be the supremum of $S$.

(b) [3pts.] Prove that if $\inf S = \sup S$, then $S$ contains exactly one element.

> **Solution:** Let $a = \sup S = \inf S$. Suppose that $s \in S$. Then since $\sup S$ is an upper bound for $S$, $s \leq \sup S = a$. Moreover, since $\inf S$ is a lower bound for $S$, $s \geq \inf S = a$. So $s = a$. Since $s$ was an arbitrary element of $S$, $S$ has exactly one element.