

MTH 320, Section 003
Analysis

Midterm 1

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books or notes. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: _____

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1.

- (a) [5pts.] List the nine defining axioms of a field F .

Solution: Let a, b, c be arbitrary elements of a field with operations $+$ and \times .

(A1) Associativity: $(a + b) + c = a + (b + c)$.

(A2) Commutativity: $a + b = b + a$.

(A3) Identity: There is an element $0 \in F$ such that $a + 0 = a$ for all a .

(A4) Inverses: For each a , there is an element $-a$ such that $a + -a = 0$.

(M1) Associativity: $(ab)c = a(bc)$.

(M2) Commutativity: $ab = ba$.

(M3) Identity: There is an element $1 \in F$ such that $a(1) = a$ for all a .

(M4) Inverses: For each $a \neq 0$, there is an element a^{-1} such that $a(a^{-1}) = 1$.

(DL) Distributive Law: $a(b + c) = ab + ac$.

- (b) [5pts.] Suppose you are given a three-element set $F = \{0, 1, a\}$ and told it has the structure of a field; that is, it has an operation $+$ and an operation \times that satisfy the axioms above. Decide what $a \times a$ is. [Hint: Because fields are closed under multiplication, your answer should be one of the three elements listed. It may be helpful to first decide which of the three elements is the multiplicative inverse of a .]

Solution: By axiom (M4), a must have a multiplicative inverse in F , i.e., there must be an element a^{-1} of F such that $a^{-1} \times a = 1$. But $a(1) = a$ by (M3) and $a(0) = 0$ by the additional properties of fields proved in class. The only element left to be the multiplicative inverse of a is a . Ergo we must have $a \times a = 1$.

Problem 2.

Let $S \subset \mathbb{R}$ be a nonempty bounded subset of \mathbb{R} .

- (a) [5pts.] Define the supremum and infimum of S .

Solution: We say b is the supremum of S if b is an upper bound for S and, if M is any upper bound for S , then $b \leq M$. Similarly, we say c is the infimum of S if c is a lower bound for S and if m is any lower bound for S , then $m \leq c$.

- (b) [5pts.] Let S and T be nonempty subsets of the \mathbb{R} with the property that $s \leq t$ for all $s \in S$ and $t \in T$. Prove that $\sup S \leq \inf T$. Is it possible for $S \cap T$ to be nonempty?

Solution: Notice that given any $t \in T$, we have $s \leq t$ for all $s \in S$, so t is an upper bound for S . Therefore, $\sup S \leq t$. This implies that in fact $\sup S$ is a lower bound for T , and therefore $\sup S \leq \inf T$. It is possible for $S \cap T$ to be nonempty; it can be a single element which is the maximum of S and minimum of T . For example, we could take $S = \{2, 3\}$ and $T = \{3, 4\}$.

Problem 3.

- (a) [5pts.] What does it mean for a sequence to be bounded?

Solution: A sequence is bounded if the set $\{s_n : n \in \mathbb{N}\}$ is bounded, or equivalently if there exists M such that $|s_n| < M$ for all n .

- (b) [5pts.] Let $(a_n), (b_n)$ be sequences of positive real numbers such that $\frac{a_n}{b_n} \rightarrow 1$. Prove that if (b_n) is bounded, then (a_n) is also bounded. [Warning: It is *not* necessarily true that $a_n < b_n$ for any particular n .]

Solution: Suppose that M is chosen such that $0 < b_n < M$ for all n . Since $\frac{a_n}{b_n} \rightarrow 1$, we know that there is some N a natural number such that for $n > N$, we have $|\frac{a_n}{b_n} - 1| < 1$. This implies that for $n > N$, $0 < \frac{a_n}{b_n} < 2$, or in particular that $0 < a_n < 2b_n < 2M$ for all $n > N$. Let $M' = \max\{a_1, \dots, a_N, 2M\}$. Then $0 < a_n < M'$ for all n . So (a_n) is bounded.

Problem 4.

Let (s_n) be a sequence defined recursively by $s_1 = 1$ and $s_{n+1} = [1 - \frac{1}{(n+1)^2}]s_n$ for $n \geq 1$.

- (a) [5pts.] Prove, using the recursive definition of (s_n) given above, that (s_n) is a convergent sequence.

Solution: Notice that, since $[1 - \frac{1}{(n+1)^2}] < 1$, we have $s_{n+1} < s_n$ for all n . Therefore (s_n) is decreasing. Moreover, $s_n \geq 0$ for all n , so (s_n) is bounded below. Since bounded monotone sequences converge, (s_n) is convergent.

- (b) [5pts.] Use induction to show that $s_n = \frac{n+1}{2n}$. What is $\lim s_n$?

Solution: For the base case, we observe that $s_1 = 1 = \frac{1+1}{2}$. For the inductive

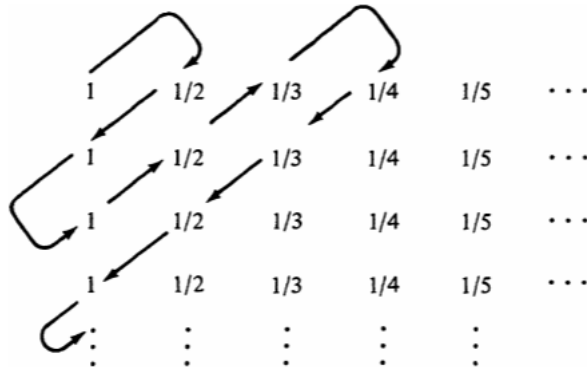
step, assume that $s_n = \frac{n+1}{2n}$. Then we have

$$\begin{aligned}
 s_{n+1} &= \left(1 - \frac{1}{(n+1)^2}\right) \left(\frac{n+1}{2n}\right) \\
 &= \frac{n+1}{2n} - \frac{1}{2n(n+1)} \\
 &= \frac{(n+1)^2 - 1}{2n(n+1)} \\
 &= \frac{n^2 + 2n}{2n(n+1)} \\
 &= \frac{n+2}{2(n+1)} \\
 &= \frac{(n+1)+1}{2(n+1)}
 \end{aligned}$$

This proves the general statement. Since $s_n = \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n}$, by the limit laws we conclude that $\lim s_n = \frac{1}{2}$.

Problem 5.

Let (s_n) be the sequence of numbers shown in the indicated order.



- (a) [5pts.] What is the set S of subsequential limits of (s_n) ?

Solution: First, we observe that the sequence is bounded, so all subsequential limits will be real numbers. Moreover, we recall that $t \in \mathbb{R}$ is a subsequential limit of (s_n) if and only if $\{n : |s_n - t| < \epsilon\}$ is infinite for all $\epsilon > 0$. We see that all numbers of the form $\frac{1}{m}$ appear infinitely many times in the sequence, hence are subsequential limits. Furthermore we note that any neighborhood of 0 of radius ϵ contains some $\frac{1}{n} < \epsilon$, which appears infinitely many times in the sequence. So 0 is also a subsequential limit. Finally, for any number t other

than 0 which is not of the form $\frac{1}{m}$ for $m \in \mathbb{N}$ there is some ϵ such that $(t-\epsilon, t+\epsilon)$ contains no terms of the form $\frac{1}{m}$ (for example, set $\epsilon = \min\{|t - \frac{1}{m}|\} > 0$. So t is not a subsequential limit of (s_n) . We conclude that $S = \{0\} \cup \{\frac{1}{m} : m \in \mathbb{N}\}$.

(b) [5pts.] What are $\limsup s_n$ and $\liminf s_n$?

Solution: $\limsup s_n = \max S = 1$, and $\liminf s_n = \min S = 0$.

This page is for scratch work. Feel free to tear it off. Do not write anything you want graded on this page unless you indicate *very clearly* that this is the case on the page of the corresponding problem.