Homework 9 Solutions

MTH 320

1. Problems from Ross.

(19.1) (a) $f(x) = x^{17}\sin x - e^x\cos(3x)$ is built by sums, products, and compositions of continuous functions, hence is continuous. Ergo since $[0,\pi]$ is a closed interval, by Theorem 19.2 f is uniformly continuous on $[0,\pi]$.

(c) $f(x) = x^3$ can be extended continuously from $(0,1)$ to $[0,1]$ by letting $f(0) = 0$ and $f(1) = 1$. Ergo by Theorem 19.5, f is uniformly continuous on $(0,1)$.

(f) If $f(x) = \sin(\frac{1}{x^2})$ on $(0,1]$, consider the Cauchy sequence (s_n) where $s_n = \frac{1}{\sqrt{\frac{n\pi}{2}}}$ for $n \geq 1$. Then $f(s_n) = \sin(\frac{n\pi}{2})$, so the sequence $(f(s_n))$ is $(1, 0, -1, 0, 1, 0, \cdots)$, which is not Cauchy. Since uniformly continuous functions map Cauchy sequences to Cauchy sequences by Theorem 19.4, f is not uniformly continuous on $(0,1]$.

(g)Let $f(x) = x^2\sin\frac{1}{x}$ on $(0,1]$. We claim that f may be extended to $\tilde{f}$ continuous on $[0,1]$ by setting $\tilde{f}(0) = 0$. For if (x_n) is any sequence in $(0,1]$ converging to 0, then $0 \leq |f(x_n)| = |x_n^2 \sin\frac{1}{x}| \leq |x_n^2|$, so since $x_n^2 \to 0$, $\tilde{f}(x_n) \to 0 = \tilde{f}(0)$. Hence $\tilde{f}$ is continuous at 0, and by Theorem 19.5 the existence of a continuous extension to $[0,1]$ suffices to show that f is uniformly continuous on $(0,1]$.

(19.2)(b) Let $f(x) = x^2$ on $[0,3]$. Let $\epsilon > 0$, and set $\delta = \frac{\epsilon}{6}$. Then for $x, y \in [0,3]$, if $|x-y| < \delta$, $|f(x) - f(y)| = |x^2 - y^2| = |x-y||x+y| < \frac{\epsilon}{6}(6) = \epsilon$. Ergo f is uniformly continuous on $[0,3]$.

(19.4)(a) Let f be uniformly continuous on a bounded set S. Suppose that f is unbounded on S. Then for any $N \in \mathbb{N}$, there is an $x_n \in S$ such that $|f(x_n)| > N$. Consider the sequence (x_n). By the Bolzano-Weierstrass Theorem, some subsequence (x_{n_k}) converges, hence is Cauchy. But by Theorem 19.4, since f is uniformly continuous, f maps Cauchy sequences to Cauchy sequences, so $(f(x_{n_k}))$ is a Cauchy sequence, hence converges to some real number. However, by construction $\lim f(x_{n_k}) = \infty$. This is a contradiction, so f must be bounded on S.

(b) Observe that $f(x) = \frac{1}{x^2}$ is not bounded on the bounded set $(0,1)$, so f cannot be uniformly continuous on $(0,1)$.

(19.5)(a) $f(x) = \tan x$ is continuous on the closed interval $S = [0\frac{\pi}{4})$, hence uniformly continuous on same.

(b) $f(x) = \tan x$ is not bounded on the bounded set $S = [0, \frac{\pi}{2})$, hence not uniformly continuous on S.

(c) Since $\lim_{x \to 0} \frac{\sin x}{x} \sin x = (1)(0) = 0$, we see that $f(x) = \frac{1}{x} \sin^2 x$ is continuous extendable to $[0, 1]$ by setting $\widetilde{f}(x) = 0$; hence f is uniformly continuous on $(0, 1]$.

(d) $f(x) = \frac{1}{x-3}$ is unbounded on the bounded set $(0, 3)$, hence not uniformly continuous on $(0, 3)$.

(e)$f(x) = \frac{1}{x-3}$ is unbounded on the bounded set $(3, 4)$, hence not uniformly continuous on $(3, 4)$. Therefore it cannot be uniformly continuous on $(3, \infty)$.

(f) Consider $f(x) = \frac{1}{x-3}$ on $(4, \infty)$. Let $\epsilon > 0$, and let $\delta = \epsilon$. Then if $x, y > 4$ and $|x - y| < \delta = \epsilon$, we have

$$|f(x) - f(y)| = \frac{|y - x|}{(x - 3)(y - 3)} < |y - x| < \delta = \epsilon.$$

So $f(x)$ is uniformly continuous on $(4, \infty)$.

(19.7) (a) Since f is continuous on $[0, \infty)$, f is continuous on $[0, k + 1]$, implying that f is in fact uniformly continuous on $[0, k + 1]$. Given $\epsilon > 0$, choose $\delta_1 < 1$ such that for $x, y \in [0, k + 1]$, $|x - y| < \delta_1$ implies that $|f(x) - f(y)| < \epsilon$. Furthermore, choose $\delta_2 < 1$ such that for $x, y \in [k, \infty)$, and $|x - y| < \delta_2$ implies that $|f(x) - f(y)| < \epsilon$. Then let $\delta = \min\{\delta_1, \delta_2\}$, and let $|x - y| < \delta$ for some $x, y \in [0, \infty)$. Because $\delta < 1$, either $x, y \in [0, k+1]$ or $x, y \in [k, \infty)$, and $|x-y| < \delta$ then implies that $|f(x)-f(y)| < \epsilon$.

(b) We claim that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, 1]$. For given $\epsilon > 0$, let $\delta = \epsilon$. Then if $x, y \in [1, \infty)$ and $|x - y| < \delta = \epsilon$, we see

$$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq |x - y| < \epsilon.$$

Hence by part (a), $\sqrt{x}$ is uniformly continuous on $[0, \infty)$.

2. (20.6) Let $f(x) = x^3|x|$. Then when $x \geq 0$, $f(x) = x^2$, and when $x < 0$, $f(x) = -x^2$. Observe that if (x_n) is any sequence of points approaching ∞, the terms of the sequence $(f(x_n)) = (x_n^2)$ also diverge to ∞, so $\lim_{x \to \infty} f(x) = \infty$. Similarly $\lim_{x \to -\infty} f(x) = -\infty$. Finally, let (x_n) be an arbitrary sequence of real numbers in $\mathbb{R}$ $\{0\}$ such that $x_n \to 0$. Then $|f(x_n)| = x_n^2$, so by the squeeze theorem, since $x_n^2 \to 0$, $f(x_n)^2 \to 0$.

Since (x_n) was arbitrary, we have proved $\lim_{x\to 0} f(x) = 0$ (and hence the left- and right-hand limits are both zero).

3. (20.13) Recall that $\lim_{x\to a} f(x) = 3$ and $\lim_{x\to a} g(x) = 2$.

   (a) By the limit laws for functions concerning multiplication by a constant and products of functions, $\lim_{x\to a} 3f(x) = 9$ and $\lim_{x\to a} g(x)^2 = 4$. Ergo $\lim_{x\to a}[3f(x)+g(x)^2] = 9+4 = 13$ since both limits in the sum exist.

   (b) By the limit law for quotients, $\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{\lim_{x\to a} g(x)} = \frac{1}{2}$.

   (c) Recall that $\sqrt{x}$ is a continuous function, so

$$\begin{aligned}\lim_{x\to a} \sqrt{3f(x)+8g(x)} &= \sqrt{\lim_{x\to a} 3f(x)+8g(x)}\\ &= \sqrt{3(3)+8(2)}\\ &= \sqrt{25}\\ &= 5\end{aligned}$$

   where the third step again follows by the limit laws for functions.

4. (20.16) (a) Let (x_n) be any sequence of points in (a,b) such that $x_n \to a$. Since $\lim_{x\to a^+} f_1(x) = L_1$, by definition $f_1(x_n) \to L_1$, and since $\lim_{x\to a^+} f_1(x) = L_1$, by definition $f_2(x_n) \to L_2$. But for all n, $f_1(x_n) \leq f_2(x_n)$, so we must have $\lim f_1(x_n) \leq \lim f_2(x_n)$. Ergo $L_1 \leq L_2$.

   (b) No. Consider $f(x) = 0$ and $f(x) = \frac{1}{x}$ on $(0,1)$.

5. (20.17) Let (x_n) be any sequence of points in (a,b) such that $x_n \to a$. Then, since $\lim_{x\to a^+} f_1(x) = L = \lim_{x\to a^+} f_3(x)$, by definition the limits of the sequences $(f_1(x_n))$ and $(f_3(x_n))$ are both L. However, $f_1(x_n) \leq f_2(x_n) \leq f_3(x_n)$ for all n, so by the Squeeze Theorem for sequences, we must have $\lim f_2(x_n) = L$. Since (x_n) was an arbitrary sequence in (a,b), this implies that $\lim_{x\to a^+} f_2(x) = L$.