

# Homework 8 Solutions

MTH 320

## 1. Problems from Ross.

(17.2)(a) The functions are as follows, and all have domain  $\mathbb{R}$ .

$$f + g(x) = \begin{cases} x^2 & \text{when } x < 0 \\ x^2 + 4 & \text{when } x \geq 0 \end{cases}$$

$$fg(x) = \begin{cases} 0 & \text{when } x < 0 \\ 4x^2 & \text{when } x \geq 0 \end{cases}$$

$$f \circ g(x) = 4$$

$$g \circ f(x) = \begin{cases} 0 & \text{when } x < 0 \\ 16 & \text{when } x \geq 0 \end{cases}$$

(b) The functions  $g$ ,  $fg$ , and  $f \circ g$  are continuous, and  $f$ ,  $f + g$ , and  $g \circ f$  are not. Sample proof: We claim  $f$  fails to be continuous at 0. If  $(x_n)$  is a sequence of points such that  $x_n < 0$  for all  $n$  and  $x_n \rightarrow 0$ , then  $\lim f(x_n) = \lim 0 = 0 \neq 4 = f(0)$ .

(17.3) (a) By assumption,  $\cos(x)$  is continuous, so since products of continuous functions are continuous,  $\cos^4 x$  is continuous. Moreover, constant functions are continuous, and sums of continuous functions are continuous, so  $1 + \cos^4 x$  is continuous. Finally,  $\log x$  is continuous on its domain, all positive numbers, so given that  $1 + \cos^4 x > 1$  and compositions of continuous functions are continuous,  $\log(1 + \cos^4 x)$  is continuous on  $\mathbb{R}$ . (b) Because  $2^x$  and  $x^2$  are continuous functions and compositions of continuous functions are continuous,  $2^{x^2}$  is also continuous.

(17.10) In each case, it suffices to find a sequence  $s_n \rightarrow x_0$  such that  $\lim f(s_n) \neq f(x_0)$ .

(a) Let  $f(x) = 1$  for  $x > 0$  and  $f(x) = 0$  for  $x \leq 0$ . Consider the sequence  $(s_n)$  where  $s_n = \frac{1}{n}$ . Then  $\lim s_n = 0$ , but  $\lim f(s_n) = \lim f(\frac{1}{n}) = 1 \neq 0$ . So  $f$  is not continuous at 0.

(b) Let  $g(x) = \sin(\frac{1}{x})$  for  $x \neq 0$  and  $g(0) = 0$ . Consider the sequence  $(s_n)$  where  $s_n = \frac{1}{(2n + \frac{1}{2})\pi}$ . Then  $\lim s_n = 0$ , but  $\lim f(s_n) = \lim \sin((2n + \frac{1}{2})\pi) = \lim 1 = 1 \neq f(0)$ . Ergo  $f$  is not continuous at 0.

(c) Let  $\text{sgn}(x) = -1$  for  $x < 0$ ,  $\text{sgn}(x) = 1$  for  $x > 0$ , and  $\text{sgn}(0) = 0$ . Again consider the sequence  $(s_n)$  such that  $s_n = \frac{1}{n}$ . Then  $\lim s_n = 0$ , but  $\lim f(s_n) = \lim 1 = 1$ . So  $f$

is not continuous at 0.

(17.12) (a) Suppose  $f$  is a continuous real-valued function on  $(a, b)$  such that  $f(r) = 0$  for all  $r$  rational in  $(a, b)$ . Let  $x \in (a, b)$ . For each  $n \in \mathbb{N}$ , choose a rational number  $r_n$  in the interval  $(x - \frac{1}{n}, x) \cap (a, b)$ . (We know this is possible because every interval contains a rational number.) Then  $|x - r_n| < \frac{1}{n}$ , so  $\lim r_n = x$ . Therefore by continuity,  $f(x) = \lim f(r_n) = \lim 0 = 0$ .

(b) Consider the function  $f - g(x)$  on  $(a, b)$ , which is continuous on  $(a, b)$  since both  $f$  and  $g$  are continuous on  $a, b$ ). Since  $f(r) = g(r)$  on all rational  $r \in (a, b)$ ,  $f - g(r) = 0$  on all rational  $r \in (a, b)$ . Ergo by part (a),  $f - g(x) = 0$  for all  $x \in (a, b)$ , so  $f(x) = g(x)$  for all  $x \in (a, b)$ .

Ergo continuous functions on intervals are determined by their values on the rational numbers!

(18.4) Let  $S \subset \mathbb{R}$ . Suppose there exists a sequence  $(x_n)$  in  $S$  such that  $x_n \rightarrow x_0 \notin S$ . Let  $f(x) = \frac{1}{x - x_0}$ . Then  $f$  is well-defined on  $S$  since  $x_0 \notin S$ , and is continuous since it is a quotient of continuous functions such that the denominator is nonzero. Now for any  $M > 0$ , choose  $N$  such that  $n > N$  implies  $|x_n - x_0| < \frac{1}{M}$ . Then for  $n > N$ ,  $|f(x_n)| = \frac{1}{|x_n - x_0|} > M$ . Since  $M$  was arbitrary,  $f$  is unbounded on  $S$ .

So any set which is not closed (i.e. does not contain all its limit points) is the domain of some unbounded continuous function.

(18.7) Let  $f(x) = xe^x$ . Since products of continuous functions are continuous,  $f$  is continuous on  $\mathbb{R}$ . Observe that  $f(0) = 0$  and  $f(1) = e \approx 2.718$ . Since  $f(0) < 2 < f(1)$ , by the Intermediate Value Theorem, there is some  $x$  in  $(0, 1)$  such that  $f(x) = 2$ .

(18.10) Let  $f$  be a continuous function on  $[0, 2]$  such that  $f(0) = f(2)$ . Consider the function  $g(x) = f(x+1) - f(x)$  on  $[0, 1]$ . Observe that  $f(x+1)$  is a composition of the continuous functions  $f(x)$  and  $x+1$ , hence continuous, so  $g$  is a difference of continuous functions and therefore continuous. Moreover,  $g(0) = f(1) - f(0) = f(1) - f(2) = -[f(2) - f(1)] = -g(1)$ . Since  $g(0) = -g(1)$ , either  $g(0) \leq 0 \leq g(1)$  or  $g(0) \geq 0 \geq g(1)$ ; in either case, by the Intermediate Value Theorem, there exists  $x \in [0, 1]$  such that  $g(x) = 0$ . This implies that  $0 = f(x+1) - f(x)$ , or equivalently  $f(x+1) = f(x)$ . Let  $y = x+1$ , then  $x, y$  have the property that  $|y - x| = 1$  and  $f(x) = f(y)$ .

Ergo if you start a car, drive for two hours, and then stop, at some point during the second hour you will be driving exactly the speed you were driving an hour ago.

2. We observe that  $(s_n^+)$  is monotone increasing, so either  $s_n^+ \rightarrow \infty$  or  $s_n^+ \rightarrow c$  for some  $c \in \mathbb{R}_{>0}$ . Suppose that  $s_n^+ \rightarrow c$ . Let  $(s_n)$  be the sequence of partial sums of  $\sum a_n = A$ , so that  $s_n \rightarrow A$ . Then  $s_n^- = s_n - s_n^+ \rightarrow A - c \in \mathbb{R}$  by the limit laws. So  $(s_n^-)$  also converges to a real number  $b = A - c$ .

Now consider the series  $\sum |a_n|$ . Let  $(t_n)$  be the sequence of partial sums of this series. Then  $t_n = s_n^+ - s_n^- \rightarrow c - b$  by the limit laws. This implies that  $\sum |a_n|$  converges, so  $\sum a_n$  converges absolutely. This is a contradiction! So in fact  $s_n^+ \rightarrow \infty$ . Similarly  $s_n^- \rightarrow \infty$ .

3. The stars over Babylon function.

- Let  $x_0 = \frac{p}{q}$  be rational, such that  $f(x_0) = \frac{1}{q}$ . Then for every  $n \in \mathbb{N}$ , choose an irrational  $x_n \in (x_0 - \frac{1}{n}, x_0) \cap (0, 1]$ . The sequence  $(x_n)$  has the property that  $|x_0 - x_n| < \frac{1}{n}$  for all  $n \in \mathbb{N}$ , so  $x_n \rightarrow x_0$ . However, since  $x_n$  is irrational,  $f(x_n) = 0$  for all  $n$ , so  $\lim f(x_n) = 0 \neq \frac{1}{q} = f(x_0)$ . So  $f$  is discontinuous at  $x_0$ .
- Let  $x_0$  be irrational, so that  $f(x_0) = 0$ . Observe that the set of values our function  $f$  takes is  $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ . Notice that for every  $n \in \mathbb{N}$ , if  $r \in (0, 1]$  has  $f(r) = \frac{1}{n}$ , it must be the case that  $r$  can be written as  $\frac{i}{n}$  for some  $i$ . Therefore, if we let

$$\delta_N = \min\{|x_0 - \frac{i}{n}| : 0 \leq i \leq n \leq N, i, n \in \mathbb{N}\},$$

we see that for any  $n \leq N$ ,  $(x_0 - \delta_N, x_0 + \delta_N)$  contains no  $r$  such that  $f(r) = \frac{1}{n}$ . Ergo  $|x - x_0| < \delta_N$  implies that  $|f(x) - f(x_0)| < \frac{1}{N}$ . Hence  $f$  is continuous at  $x_0$ .