MTH 320: Homework 8

Due: October 27, 2017

1. Note that Quiz 2 will take place on Monday October 30, during the first twenty minutes of class. Material that is fair game is Sections 14-15, 17-18, plus rearrangements of series. (Section 18 will conclude sometime during lecture on Monday October 23.)

2. Read Sections 19-20 in Ross.

3. Do problems 17.2, 17.3 (a),(c), 17.10, 17.12, 18.4, 18.7, 18.10 in Ross.

4. Prove the following lemma mentioned in class: Let $\sum_{n=1}^{\infty} a_n$ be a conditionally convergent series, and set

$$a_n^+ = \begin{cases} a_n & a_n \geq 0 \\ 0 & a_n < 0 \end{cases}$$

$$a_n^- = \begin{cases} a_n & a_n \leq 0 \\ 0 & a_n > 0. \end{cases}$$

   Let (s_n^+), (s_n^-) be the sequences of partial sums of $\sum a_n^+$ and $\sum a_n^-$. Then $s_n^+ \to \infty$ and $s_n^- \to -\infty$.

5. *The stars over Babylon.* For each rational number $r \in (0,1]$, write $r = \frac{p}{q}$ where $p, q \in \mathbb{N}$ are natural numbers with no common factors. Then consider the following function on $[0,1]$:

$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \text{ is rational} \\ 0 & x \text{ is irrational.} \end{cases}$$

   We claim that f is discontinuous at every rational number in $(0,1]$ and continuous at every rational.

   - Discontinuity at each rational. Let $x_0 \in (0,1]$ such that x_0 is rational. For $n \in \mathbb{N}$, pick x_n an irrational in $(x_0 - \frac{1}{n}, x_0) \cap (0,1]$. Use this sequence to show f is discontinuous at x_0.
   - Continuity at each irrational. Let $x_0 \in (0,1]$ such that x_0 is irrational. Let N be a natural number. Let

$$\delta_N = \min\{|x_0 - \frac{i}{n}| : 0 \leq i \leq n \leq N,\ i, n \in \mathbb{N}\}.$$

     Observe that because $x_0 \neq \frac{i}{n}$ for any $\frac{i}{n}$, $\delta_N > 0$. Prove that for $x \in (0,1]$, if $|x - x_0| < \delta_N$, then $|f(x) - f(x_0)| < \frac{1}{N}$. Conclude that f is continuous at x_0.

   This example helps demonstrate that our intuition for what continuity should "look like" on a graph is in general insufficiently subtle.