Homework 7 Solutions

MTH 320

1. Problems from Section 14.

   (14.2) (d) We note that if $a_n = \frac{n^3}{3^n}$, then $\lim |\frac{a_{n+1}}{a_n}| = \lim \frac{(n+1)^3}{3n^3} = \frac{1}{3}$. So by the Ratio Test this series converges.

   (14.2) (e) We note that if $a_n = \frac{n^2}{n!}$, then $\lim |\frac{a_{n+1}}{a_n}| = \lim \frac{(n+1)}{n^2} = 0$. So by the Ratio Test this series converges.

   (14.2)(f)$\sum \frac{1}{n^n}$. We use the Root Test, observing that $\limsup |\frac{1}{n^n}|^{\frac{1}{n}} = \limsup \frac{1}{n} = 0$. Ergo this series converges.

   (14.6) (a) Suppose that $\sum |a_n|$ converges and (b_n) is bounded. Pick M such that $|b_n| < M$ for all n. Then $\sum M|a_n|$ converges, and $|a_n b_n| < M|a_n|$ for all n, so $\sum a_n b_n$ converges by the Comparison Test.

   (b) Corollary 14.7 follows via setting $b_n = 1$ for all n.

   (14.12) Let (a_n) be a sequence with $\liminf |a_n| = 0$. We will use induction to construct a subsequence (a_{n_k}) such that $\sum_{k=1}^{\infty} a_{n_k}$ converges by comparison with $\sum \frac{1}{k^2}$. Since $\liminf |a_n| = 0$, there is some N_1 such that $n \geq N_1$ implies $|a_n| < 1 = \frac{1}{1^2}$. Let $a_{n_1} = a_{N_1+1}$. Now, in general suppose that we have chosen $n_1 < n_2 < \cdots < n_{k-1}$ such that $|a_{n_\ell}| < \frac{1}{\ell^2}$ for all $1 \leq \ell \leq k-1$. There is some N_k such that $n > N_k$ implies $|a_n| \leq \frac{1}{k^2}$. Let $n_k = \max\{N_k, n_{k-1}\} + 1$. Then $n_k \geq n_{k-1}$ and $|a_{n_k}| < \frac{1}{k^2}$. This gives us a sequence (a_{n_k}) such that $|a_{n_k}| < \frac{1}{k^2}$ for all $k \geq 1$, so $\sum_{k=1}^{\infty} a_{n_k}$ converges by the Comparison Test.

2. Problems from Section 15.

   (15.1) (a) As in class, we see that $\frac{1}{n}$ is decreasing to zero, so the alternating series converges.

   (b) Let $a_n = \frac{(-1)^n n!}{2^n}$. Then $|\frac{a_{n+1}}{a_n}| = \frac{n+1}{2}$, so $\liminf |\frac{a_{n+1}}{a_n}| = \infty$, and the series $\sum a_n$ diverges by the Ratio Test.

   (15.4) (b) Let $a_n = \frac{\log n}{n}$. Then if $f(x) = \frac{\log x}{x}$, $a_n = f(n)$. We see that f is positive, and since $f'(x) = \frac{1}{x^2} - \frac{\log x}{x^2} < 0$ for $x > e$, f is eventually decreasing. We may apply the Integral Test:

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{\log n}{n} &\geq \lim_{t\to\infty} \int_2^t \frac{\log x}{x} dx \\
&= \lim_{t\to\infty} \left[\frac{(\log x)^2}{2} \right]_2^t \\
&= \lim_{t\to\infty} \left(\frac{(\log t)^2}{2} - \frac{(\log 2)^2}{2} \right) \\
&= \infty
\end{aligned}$$

We conclude that the series $\sum_{n=2}^{\infty} \frac{\log n}{n}$ diverges.

(d) Let $a_n = \frac{\log n}{n^2}$ Then if $f(x) = \frac{\log x}{x^2}$, $a_n = f(n)$. We see f is positive, and $f'(x) = \frac{1}{x^3} - \frac{2 \log x}{x^3} < 0$ for $x \geq 2$, f is eventually decreasing. We may apply the Integral Test:

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{\log n}{n^2} &\geq \lim_{n} \int_2^n \frac{\log x}{x^2} dx \\
&= \lim_{t\to\infty} \left[-\frac{\log x - 1}{x} \right]_2^t \\
&= \lim_{t\to\infty} \left(\frac{1 - \log t}{t} + \frac{\log 2 - 1}{2} \right) \\
&= \frac{\log 2 - 1}{2}
\end{aligned}$$

We conclude that the series $\sum \frac{\log n}{n^2}$ converges.

3. The number e.

- (a) Let $b_n = \frac{1}{n!}$. Then $|\frac{b_n+1}{b_n}| = |\frac{1}{n+1}|$, so $\limsup |\frac{b_n+1}{b_n}| = 0$. Ergo $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

- (b) We compute that

$$\begin{aligned}
a_n &= (1 + \frac{1}{n})^n \\
&= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \left(\frac{1}{n}\right)^k \\
&= \sum_{k=0}^{n} \frac{n!}{(n-k)!} \left(\frac{1}{n}\right)^k \frac{1}{k!} \\
&= \frac{1}{0!} + \sum_{k=1}^{n} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{1}{k!}
\end{aligned}$$

Observe that for $n \geq 1$, $\frac{n(n-1)\cdots(n-k+1)}{n^k} \leq 1$, so $a_n \leq \frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{n!} = s_n$. Therefore $\limsup a_n \leq \limsup s_n = \lim s_n = s$.

- (c) Notice that

$$\begin{aligned} a_n &= \frac{1}{0!} + \sum_{k=1}^{n} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{1}{k!} \\ &= \frac{1}{0!} + \sum_{k=1}^{m} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{1}{k!} + \sum_{k=m+1}^{n} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{1}{k!} \\ &\leq \frac{1}{0!} + \sum_{k=1}^{m} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{1}{k!} \end{aligned}$$

  Letting $n \to \infty$ as indicated, we see $\liminf a_n \geq \lim \frac{1}{0!} + \sum_{k=1}^{m} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{1}{k!} = \sum_{k=0}^{m} \frac{1}{m!}$. But m was arbitary, so in fact $\liminf a_n \geq \lim \sum_{k=0}^{m} \frac{1}{m!} = s$.

- (d) Since $\limsup a_n = \liminf a_n = s$, $\lim a_n = s$. SO both these possible definitions of e are the same.

This outline is based on the proof given in Rudin's book *Principles of Mathematical Analysis*.