1. Problems from Section 14.

(14.2) (d) We note that if \(a_n = \frac{n^3}{3^n}\), then \(\lim \frac{|a_{n+1}|}{a_n} = \lim \frac{(n+1)^3}{3^{n+1}} = \frac{1}{3}\). So by the Ratio Test this series converges.

(14.2) (e) We note that if \(a_n = \frac{n^2}{n!}\), then \(\lim \frac{|a_{n+1}|}{a_n} = \lim \frac{(n+1)^2}{n!} = 0\). So by the Ratio Test this series converges.

(14.2) (f) \(\sum \frac{1}{n^n}\). We use the Root Test, observing that \(\lim \sup \frac{1}{n^n} = \lim \sup \frac{1}{n} = 0\). Ergo this series converges.

(14.6) (a) Suppose that \(\sum |a_n|\) converges and \((b_n)\) is bounded. Pick \(M\) such that \(|b_n| < M\) for all \(n\). Then \(\sum M|a_n|\) converges, and \(|a_n b_n| < M|a_n|\) for all \(n\), so \(\sum a_n b_n\) converges by the Comparison Test.

(b) Corollary 14.7 follows via setting \(b_n = 1\) for all \(n\).

(14.12) Let \((a_n)\) be a sequence with \(\lim \inf |a_n| = 0\). We will use induction to construct a subsequence \((a_{n_k})\) such that \(\sum_{k=1}^{\infty} a_{n_k}\) converges by comparison with \(\sum_{k=1}^{\infty} \frac{1}{k^2}\). Since \(\lim \inf |a_n| = 0\), there is some \(N_1\) such that \(n \geq N_1\) implies \(|a_n| < 1 = \frac{1}{2^2}\). Let \(a_{n_1} = a_{N_1+1}\). Now, in general suppose that we have chosen \(n_1 < n_2 < \cdots < n_{k-1}\) such that \(|a_{n_{\ell}}| < \frac{1}{k^2}\) for all \(1 \leq \ell \leq k - 1\). There is some \(N_k\) such that \(n > N_k\) implies \(|a_n| \leq \frac{1}{k^2}\). Let \(n_k = \max\{N_k, n_k-1\} + 1\). Then \(n_k \geq n_{k-1}\) and \(|a_{n_k}| < \frac{1}{k^2}\). This gives us a sequence \((a_{n_k})\) such that \(|a_{n_k}| < \frac{1}{k^2}\) for all \(k \geq 1\), so \(\sum_{k=1}^{\infty} a_{n_k}\) converges by the Comparison Test.

2. Problems from Section 15.

(15.1) (a) As in class, we see that \(\frac{1}{n}\) is decreasing to zero, so the alternating series converges.

(b) Let \(a_n = \frac{(-1)^n n!}{2^n}\). Then \(\frac{|a_{n+1}|}{a_n} = \frac{n+1}{2}\), so \(\lim \inf \frac{|a_{n+1}|}{a_n} = \infty\), and the series \(\sum a_n\) diverges by the Ratio Test.

(15.4) (b) Let \(a_n = \frac{\log n}{n}\). Then if \(f(x) = \frac{\log x}{x}\), \(a_n = f(n)\). We see that \(f\) is positive, and since \(f'(x) = \frac{1}{x^2} - \frac{\log x}{x^2} < 0\) for \(x > e\), \(f\) is eventually decreasing. We may apply
the Integral Test:

$$\sum_{n=2}^{\infty} \frac{\log n}{n} \geq \lim_{t \to \infty} \int_{2}^{t} \frac{\log x}{x} \, dx$$

$$= \lim_{t \to \infty} \left[ \frac{(\log x)^2}{2} \right]_{2}^{t}$$

$$= \lim_{t \to \infty} \left( \frac{(\log t)^2}{2} - \frac{(\log 2)^2}{2} \right)$$

$$= \infty$$

We conclude that the series $$\sum_{n=2}^{\infty} \frac{\log n}{n}$$ diverges.

(d) Let $$a_n = \frac{\log n}{n^2}$$. Then if $$f(x) = \frac{\log x}{x^2}$$, $$a_n = f(n)$$. We see $$f$$ is positive, and $$f'(x) = \frac{1}{x} - \frac{2\log x}{x^2} < 0$$ for $$x \geq 2$$, $$f$$ is eventually decreasing. We may apply the Integral Test:

$$\sum_{n=2}^{\infty} \frac{\log n}{n^2} \geq \lim_{n \to \infty} \int_{2}^{n} \frac{\log x}{x^2} \, dx$$

$$= \lim_{t \to \infty} \left[ -\frac{\log x - 1}{x} \right]_{2}^{t}$$

$$= \lim_{t \to \infty} \left( \frac{1 - \log t}{t} + \frac{\log 2 - 1}{2} \right)$$

$$= \frac{\log 2 - 1}{2}$$

We conclude that the series $$\sum_{n=2}^{\infty} \frac{\log n}{n^2}$$ converges.

3. The number $$e$$.

- (a) Let $$b_n = \frac{1}{n!}$$. Then $$\left| \frac{b_{n+1}}{b_n} \right| = \frac{1}{n+1}$$, so $$\lim \sup \left| \frac{b_{n+1}}{b_n} \right| = 0$$. Ergo $$\sum_{n=0}^{\infty} \frac{1}{n!}$$ converges.

- (b) We compute that

$$a_n = (1 + \frac{1}{n})^n$$

$$= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \left( \frac{1}{n} \right)^k$$

$$= \sum_{k=0}^{n} \frac{n!}{(n-k)!} \left( \frac{1}{n} \right)^k \frac{1}{k!}$$

$$= \frac{1}{0!} + \sum_{k=1}^{n} \frac{n(n-1) \cdots (n-k+1) 1}{n^k} \frac{1}{k!}$$
Observe that for $n \geq 1$, $\frac{n(n-1) \cdots (n-k+1)}{n^k} \leq 1$, so $a_n \leq \frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{n!} = s_n$. Therefore $\limsup a_n \leq \limsup s_n = \lim s_n = s$.

- (c) Notice that

$$a_n = \frac{1}{0!} + \sum_{k=1}^{n} \frac{n(n-1) \cdots (n-k+1)}{n^k} \frac{1}{k!}$$

$$= \frac{1}{0!} + \sum_{k=1}^{m} \frac{n(n-1) \cdots (n-k+1)}{n^k} \frac{1}{k!} + \sum_{k=m+1}^{n} \frac{n(n-1) \cdots (n-k+1)}{n^k} \frac{1}{k!}$$

$$\leq \frac{1}{0!} + \sum_{k=1}^{m} \frac{n(n-1) \cdots (n-k+1)}{n^k} \frac{1}{k!}$$

Letting $n \to \infty$ as indicated, we see $\liminf a_n \geq \lim \frac{1}{0!} + \sum_{k=1}^{m} \frac{n(n-1) \cdots (n-k+1)}{n^k} \frac{1}{k!} = \sum_{k=0}^{m} \frac{1}{m!}$. But $m$ was arbitrary, so in fact $\liminf a_n \geq \lim \sum_{k=0}^{m} \frac{1}{m!} = s$.

- (d) Since $\limsup a_n = \liminf a_n = s$, $\lim a_n = s$. SO both these possible definitions of $e$ are the same.

This outline is based on the proof given in Rudin’s book *Principles of Mathematical Analysis*. 