MTH 320: Homework 7

Due: October 20, 2017

1. Read Sections 17-18 in Ross.
2. Do problems 14.2 (d), (e), (f), 14.6, 14.12 in Ross.
3. Do problems 15.1, 15.4(b), (d), in Ross. [You can use whatever statements you like from your calculus class to do 15.4.]
4. *The number e.* You have probably seen in calculus that Euler's number e may be defined as the limit of the sequence $a_n = (1 + \frac{1}{n})^n$. This is sometimes described as the interaction between the "irresistible force" – to wit, an exponent approaching infinity – and the "immovable object" – to wit, a base approaching 1. Another possible definition of e is

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

   We will show these expressions are both convergent, and in fact coincide. Let s_n be the partial sums of the series $\sum_{n=0}^{\infty} \frac{1}{n!}$.

   - (a) Show that $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges. Call the limit s.
   - (b) The binomial theorem states that, for $n \geq 1$, $(1+x)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k$. With this in mind, show that for $n \geq 1$,

$$a_n = \frac{1}{0!} + \sum_{k=1}^{n} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{1}{k!}$$

     Conclude that $a_n \leq s_n$ for all $n \geq 1$, and therefore $\limsup a_n \leq s$.
   - (c) For $n \geq m$, show that

$$a_n \geq \frac{1}{0!} + \sum_{k=1}^{m} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{1}{k!}.$$

     Letting $n \to \infty$ for fixed m, observe that we have $\liminf a_n \geq 1 + 1 + \frac{1}{2!} + \cdots \frac{1}{m!}$. Since m was arbitrary, conclude that $\liminf a_n \geq s$.
   - (d) From the above, conclude that (a_n) converges and $\lim a_n = s$. Therefore the two definitions of e above are convergent and equal.