Homework 6 Solutions

MTH 320

1. Problems from Section 14.

   (14.2) (a) $\sum \frac{(n-1)}{n^2}$. Observe that $\frac{(n-1)}{n^2} = \frac{1}{n}(1 - \frac{1}{n}) \geq \frac{1}{2n}$ for $n \geq 2$. Ergo this series diverges by the Comparison Test.

   (14.3)(b) $\sum \frac{2+\cos n}{3^n}$. Observe that $0 < \frac{2+\cos n}{3^n} \leq \frac{3}{3^n} = \frac{1}{3^{n-1}}$, which is geometric. Therefore this series converges by the Comparison Test.

   (14.3)(e)$\sum \frac{n\pi}{9}$. Observe that the sequence $a_n = \frac{n\pi}{9}$ has a subsequence $(a_3, a_{21}, a39, \cdots) = (\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, \cdots)$ consisting of all terms of the form a_{18m+3} which is constant at $\frac{\sqrt{3}}{2}$. Ergo $\lim a_n \neq 0$, so the series must diverge.

   (14.4)(c) $\sum \frac{1}{2^n+n}$. Observe that $0 < \sum \frac{1}{2^n+n} < \frac{1}{2^n}$, which is geometric. Therefore this series converges by the Comparison Test.

2. (14.5)Let $\sum a_n = A$ and $\sum b_n = B$. Without loss of generality both series start at $n = 1$. Let $s_n = a_1 + \cdots a_n$ be the partial sums of (a_n) and $t_n = t_1 + \cdots t_n$ be the partial sums of (b_n). Then $\lim s_n = A$, $\lim t_n = B$.

   (a) The partial sums of $(a_n + b_n)$ are $s_n + t_n$. Therefore $\sum(a_n + b_n) = \lim(s_n + t_n) = \lim s_n + \lim t_n = A + B$ because both (s_n) and (t_n) converge.

   (b)The partial sums of $(ka_n$ are ks_n. Therefore $\sum ka_n = \lim k(s_n) = k \lim s_n = kA$.

   (c) No. For consider the sequences

$$(a_n) = \left(\frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{8}, 0, \cdots\right)$$
$$(b_n) = \left(0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{8}, \cdots\right).$$

   By our study of geometric series in class, $\sum a_n = 1 = \sum b_n$. So $AB = 1$. However, $(a_nb_n) = (0, 0, 0, 0, 0, \cdots)$ and therefore $\sum a_nb_n = 0$.

   (14.8) Let $\sum a_n$ and $\sum b_n$ be convergent series of nonnegative numbers. Then consider the square $(\sqrt{a_n} - \sqrt{b_n})^2 = a_n - 2\sqrt{a_nb_n} + b_n^2$. Because squares are nonnegative, we see that $a_n - 2\sqrt{a_nb_n} + b_n^2 \geq 0$, implying that $a_n + b_n \geq 2\sqrt{a_nb_n} \geq \sqrt{a_nb_n} = |\sqrt{a_nb_n}|$. Now by Exercise 14.5 from last week, $\sum(a_n + b_n)$ converges, so by the Comparison Test,

$\sum \sqrt{a_n b_n}$ also converges.

$n_k = \max\{N_k, n_{k-1}\} + 1$. Then $n_k \geq n_{k-1}$ and $|a_{n_k}| < \frac{1}{k^2}$. This gives us a sequence (a_{n_k}) such that $|a_{n_k}| < \frac{1}{k^2}$ for all $k \geq 1$, so $\sum_{k=1}^{\infty} a_{n_k}$ converges by the Comparison Test.

(14.13) (a) Recall that $\sum_{n=0}^{\infty} r^k = \frac{1}{1-r}$ whenever $|r| < 1$. Therefore

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n &= \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n - 1 \\ &= \frac{1}{1-\frac{2}{3}} - 1 \\ &= 3 - 1 \\ &= 2. \end{aligned}$$

$$\begin{aligned} \sum_{n=1}^{\infty} \left(-\frac{2}{3}\right)^n &= \sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n - 1 \\ &= \frac{1}{1+\frac{2}{3}} - 1 \\ &= \frac{1}{2}. \end{aligned}$$

(b) Using the hint, we see that partial sums s_n of $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ are

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \frac{1}{k(k+1)} \\ &= \sum_{k=1}^{n} \left[\frac{1}{k} - \frac{1}{k+1}\right] \\ &= \left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots \left(\frac{1}{n} - \frac{1}{n+1}\right)\right] \\ &= \left[1 - \frac{1}{n+1}\right] \end{aligned}$$

So $\lim s_n = 1$, and the series sums to 1.

(c) Extremely similar to (b).

(d) Observe that when $n = 1$, $\frac{n-1}{2^n} = \frac{0}{2} = 0$, so the $n = 1$ term of the series in part (c) can be discarded. Therefore $\frac{1}{2} = \sum_{n=1}^{\infty} \frac{n-1}{2^{n-1}} = \sum_{n=2}^{\infty} \frac{n-1}{2^{n-1}} = \sum_{m=1}^{\infty} \frac{m}{2^m}$, where in the last step we reindex by $m = n - 1$. Ergo the sum $\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2}$.