Homework 5 Solutions

MTH 320

1. (11.2) We'll do all five parts for each sequence in turn.

   $a_n = (-1)^n$. Two monotone subsequences are $(x_1, x_3, x_5, \cdots) = (1, 1, 1, \cdots)$, and similarly for the even entries. The set of subsequential limits is $\{\pm 1\}$. Therefore $\limsup a_n = 1$ and $\liminf a_n = -1$. This sequence is bounded, $|a_n| < 1$ for all n, but does not converge or diverge to $\pm\infty$.

   $b_n = \frac{1}{n}$. The sequence (b_n) is monotone decreasing, so any subsequence is as well; furthermore (b_n) converges to 0, so any subsequence of (b_n) also converges to 0. Therefore the set of subsequential limits is $\{0\}$, and $\limsup b_n = \liminf b_n = 0$. Finally (b_n) is bounded, e.g. $|b_n| < 2$ for all n.

   $u_n = (-\frac{1}{2})^n$. Any subequence (x_{n_i}) where all the n_i are even is monotone decreasing, and any subsequence (x_{n_i}) where all the n_i are odd is monotone increasing. The sequence u_n converges to 0, so the set of subsequential limits is $\{0\}$, and $\limsup u_n = \liminf u_n = 0$. Finally, u_n is bounded, e.g. $|u_n| < 1$ for all n.

   $x_n = 5^{(-1)^n}$. Possible monotone subsequences include $(x_1, x_3, x_5, \cdots) = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \cdots)$ and $(x_2, x_4, x_6, \cdots) = (5, 5, 5, \cdots)$. Note that $(5, 5, \cdots, 5, \frac{1}{5}, \frac{1}{5}, \cdots)$ is also monotone decreasing, and $(\frac{1}{5}, \frac{1}{5}, \cdots \frac{1}{5}, 5, 5, 5, \cdots)$ is monotone increasing. The set of subsequential limits is $\{\frac{1}{5}, 5\}$, and $\limsup x_n = 5$ while $\liminf x_n = \frac{1}{5}$. The sequence x_n neither converges nor diverges to $\pm\infty$.

   $z_n = n\cos(\frac{n\pi}{4})$. Observe that the values of this sequence are

   $$(\frac{1}{\sqrt{2}}, 0, -\frac{3}{\sqrt{2}}, -4, -\frac{5}{\sqrt{2}}, 0, \frac{7}{\sqrt{2}}, 8, \cdots)$$

   and in general the sequence contains arbitrarily large positive numbers, arbitrarily large negative numbers, and infinitely many zeroes. We can get a monotone sequence by (for example) considering all terms of the form $n(\frac{1}{\sqrt{2}})$, the form $n\frac{-1}{\sqrt{2}}$, or of the form n, or of the form $-n$. We could also simply choose the zero subsequence. The set of subsequential limits is $\{0, \pm\infty\}$, and $\limsup z_n = \infty$, while $\liminf z_n = -\infty$. The sequence does not converge or diverge to $\pm\infty$, nor is it bounded.

2. (12.3) (a) We see that $\inf\{s_n : n > N\} = 0$ for all N, so $\liminf s_n = 0$. Similarly $\liminf t_n = 0$, so $\liminf s_n + \liminf t_n = 0$.

   (b) The sequence $(s_n + t_n)$ is the repeating sequence $(2, 2, 3, 1, 2, 2, 3, 1, \cdots)$. We see that $\inf\{s_n + t_n : n > N\} = 1$ for all N, so $\liminf(s_n + t_n) = 1$.

(c) We see that $\sup\{s_n : n > N\} = 2$ for all N, so $\limsup s_n = 2$. Therefore $\liminf t_n + \limsup s_n = 0 + 2 = 2$.

(g) The sequence $(s_n t_n)$ is the repeating sequence $(0, 1, 2, 0, 0, 1, 2, 0, \cdots)$, so $\limsup(s_n t_n) = 2$.

3. Further problems from Ross.

- (11.5)Let (q_n) be an enumeration of all the rationals in $(0, 1]$.

  (a) We claim that the set S of subsequential limits is $[0, 1]$. Recall that s is a subsequential limit of (q_n) if and only if there are infinitely many points of (q_n) in $(t - \epsilon, t + \epsilon)$ for all ϵ. This clearly cannot hold for any $t \notin [0, 1]$. For $t \in (0, 1)$, we claim that density of $\mathbb{Q}$ in $\mathbb{R}$ shows that there are infinitely many rationals in each $(t - \epsilon, t + \epsilon)$. For suppose there are only finitely many. Then we can make a list of rationals $r_1 < \cdots < r_n$ lying in $(t - \epsilon, t + \epsilon$ in ascending order. Consider the interval (r_1, r_2). Density of $\mathbb{Q}$ in $\mathbb{R}$ tells us that this interval contains at least one rational r. But r cannot be any of the r_i on our list, because $r_1 < r_i < r_2$. This is a contradiction, so there must be infinitely many rationals in $(t - \epsilon, t + \epsilon)$ (and indeed any interval in $\mathbb{R}$). The argument is the same for 0 and 1, but 0 does merit a moment's additional attention: even though 0 is not included in the sequence, there are infinitely many rationals in $(0, \epsilon) \subset (-\epsilon, \epsilon)$. A similar argument works for 1.

  (b) Recall from Theorem 11.8 that $\limsup q_n = \sup S = 1$ and $\liminf q_n = \inf S = 0$. Note that this would have been extremely annoying to compute by hand!

- (11.9b) There is no such sequence. For, recall from Theorem 11.8 that if S is the set of subsequential limits of a sequence (s_n), and t is the limit of some set of points t_n in S, then t is in S. However, all the points of the sequence $(t_n) = (\frac{1}{n})_{n=1}^{\infty}$ are in $(0, 1)$, but the limit $t = 0$ of this sequence is not.

- (12.4) Let (s_n) and (t_n) be two sequences. For any $N > 0$, if $n > N$, we have $s_n \leq \sup\{s_m : m > N\}$ and $t_n \leq \sup\{s_m : m > N\}$. Therefore for $n > N$, $s_n + t_n \leq \sup\{s_m : m > N\} + \sup\{s_m : m > N\}$. Hence $\sup\{s_n + t_n : n > N\} \leq \sup\{s_n : n > N\} + \sup\{s_n : n > N\}$. Therefore applying Exercise 9.9(c), which is proved below, the same inequality holds in the limit, i.e. $\limsup s_n + t_n = \lim(\sup\{s_n + t_n : n > N\}) \leq \lim(\sup\{s_n : m > N\} + \sup\{t_n : m > N\}) = \limsup s_n + \limsup t_n$.

  Claim(Exercise 9.9c, referenced in the hint): If (a_n) and (b_n) are two convergent sequences satisfying $a_n \leq b_n$ for all n then $\lim a_n \leq \lim b_n$. Proof: Suppose not. Then suppose $\lim a_n = a > \lim b_n = b$. Let $\epsilon = \frac{a-b}{2}$. Then there exists N_1 such that $n > N_1$ implies $a_n > a - \epsilon$, and N_2 such that $n > N_2$ implies $b_n < b + \epsilon$.

But then for $n > \max\{N_1, N_2\}$, $b_n < b + \epsilon = \frac{a+b}{2} = a - \epsilon = a_n$, a contradiction.

- (12.6) For (a), we have (s_n) a bounded sequence, and $k > 0$. If v_N is the supremum of $\{s_n : n > N\}$, by multiplicative invariance of the order relation, kv_N is the supremum of $\{ks_n : n > N\}$. Ergo $\limsup ks_n = \lim_{N\to\infty} kv_N = k\lim_{N\to\infty} v_N = k\limsup s_n$. A similar proof shows the same result for the lim inf in (b). However, multiplication by a negative number is order reversing, so if $k < 0$, kv_N is the infimum of $\{ks_n : n > N$, and therefore after taking limits, $\liminf ks_n = k\limsup ks_n$. A similar result holds for $\limsup ks_n$.

- (12.10) Suppose (s_n) is bounded, so that $|s_n| < M$ for some M. Then in particular for all N, $v_N = \sup\{|s_n| : n > N\} < M$, implying that $\limsup |s_n| \leq M < \infty$. Conversely, if $\limsup |s_n| = M < \infty$, then there exists N such that $\sup\{|s_n| : n > N\} < M + 1$, and in particular $|s_n| < M + 1$ for all $n > N$. Let $M' = \max\{|s_1|, \cdots, |s_N|, M + 1\}$. Then $|s_n| < M'$ for all n. Hence (s_n) is bounded.

4. Problem 5. Say (s_n) is a sequence. Then pick any real number v in $[0, 1)$ and consider the binary decimal expansion of v. Choose a subsequence (s_{n_k}) of (s_n) by including s_n in the subsequence if and only if the nth digit of the decimal expansion is zero. This gives a unique subsequence of (s_n) associated to v. (Note that every binary decimal expansion of a real number contains infinitely many zeroes,so this is in fact a subsequence and not a finite list.) Therefore the set of subsequences of (s_n) is uncountable: if it were possible to list all subsequences of (s_n), it would be possible to list all real numbers in $[0, 1)$.