Homework 4 Solutions

MTH 320

2. Problems from Ross.

- (9.10) (a) Suppose $s_n \to \infty$. Let $k > 0$. Then for $M > 0$, there exists N such that $n > N$ implies $s_n > \frac{M}{k}$, so $ks_n > M$. Ergo $ks_n \to \infty$.

  (b) Suppose $s_n \to \infty$. Then given $M < 0$, there exists N such that $n > N$ implies $s_n > -M$, so that $-s_n < M$. Ergo $(-s_n)$ diverges to $-\infty$. The converse is similar.

  (c)Let $s_n \to \infty$ and $k < 0$. Then by (a), $(-k)s_n \to \infty$ and therefore by (b), $-(-k)s_n = ks_n \to -\infty$.

- (9.12) (a) Assume $L < 1$. Following the hint, we choose a such that $L < a < 1$. Let $\epsilon = a - L$. Choose an integer N' such that $n > N'$ implies $||\frac{s_{n+1}}{s_n}| - L| < \epsilon$. Let $N = N' + 1$, then $n \geq N$ implies $||\frac{s_{n+1}}{s_n}| - L| < \epsilon$, or equivalently $L - \epsilon < |\frac{s_{n+1}}{s_n}| < L + \epsilon = a$. Therefore since $|\frac{s_{n+1}}{s_n}| < a$ for $n > N$, we have $|s_{n+1}| \leq a|s_n|$ for $n \geq N$. Therefore inductively $|s_n| \leq a^{n-N}|s_N|$ for all $n \geq N$. By Theorem 9.7(b), since $|a| < 1$, $\lim a^{n-N} = 0$, so since $|s_N|$ is a constant, $\lim a^{n-N}|s_N| = 0$. Ergo, by exercise 8.5(b) (or by a quick proof) $\lim s_n = 0$ as well.

  (b) Assume $L > 1$. Then consider the sequence $t_n = \frac{1}{|s_n|}$. We see that $\lim |\frac{t_{n+1}}{t_n}| = \lim |\frac{s_n}{s_{n+1}}| = \lim \frac{1}{|\frac{s_{n+1}}{s_n}|} = \frac{1}{L} < 1$. Ergo by part (a), $\lim t_n = 0$. Hence by Theorem 9.10, $\lim |s_n| = \infty$.
- (9.14) Let $s_n = \frac{a^n}{n^p}$.
  Observe that $|\frac{s_{n+1}}{s_n}| = |\frac{a^{n+1}n^p}{(n+1)^p a^n} = |a|(\frac{n}{n+1})^p$. Therefore $\lim |\frac{s_{n+1}}{s_n}| = |a| \lim(\frac{n}{n+1})^p$ We claim $\lim(\frac{n}{n+1})^p = 1$; for a proof, see below. Therefore $\lim |\frac{s_{n+1}}{s_n}| = |a|$. If $|a| \leq 1$, by (9.12) part (a), $\lim s_n = 0$. If $a > 1$, then $\lim |s_n| = \infty$, but $s_n = \frac{a^n}{n^p}$ is always positive, so in fact $\lim s_n = \infty$. If $a < -1$, it is still the case that $\lim |s_n| = \infty$, but for n even, $s_n > 0$, and for n odd, $s_n < 0$, so (s_n) has no limit.

  Proof that $\lim(\frac{n}{n+1})^p = 1$. Observe that $\frac{n}{n+1} = (1 - \frac{1}{n+1}$. Given $\epsilon < 0$, let $N = \frac{1}{1-(1+\epsilon)^{\frac{1}{p}}} - 1$. Then a computation shows that $n > N$ implies $1 - (1 - \frac{1}{n+1})^p < \epsilon$.
- (10.6) (a) We claim (s_n) is Cauchy. Let $\epsilon > 0$, and choose N such that $2^{-N-1} < \epsilon$. This is certainly possible since $2^n \to \infty$, so $2^{-n} \to 0$. Then let n, m ¿ N. Without loss of generality, $m < n$, and we have $|s_n - s_m| = |(s_n - s_{n-1}) + (s_{n-1} - s_{n-2} + \cdots (s_{m+1} - s_m)| < |(s_n - s_{n-1}| + \cdots + |s_{m+1} - s_m|$. Now observe that since $m > N$, $|s_{m+1} - s_m| < 2^{-N}$, and more generally since $m + k > N + k$, $|s_{m+k+1} - s_{m+k}| < 2^{-N-k}$. Applying this principle for $k = 0, 1, \cdots, n-1$ yields

$$
\begin{aligned}
|s_n - s_m| &< 2^{-N-k} + \cdots + 2^{-N-1} + 2^{-N} \\
&= 2^{-N-k}(1 + 2 + 2^2 + ... + 2^k) \\
&= 2^{-N-k}(2^{k+1} - 1) \\
&= 2^{-N-1} - 2^{-N-k} \\
&< 2^{-N-1} \\
&< \epsilon
\end{aligned}
$$

Here the third line is an application of an induction done in class during the first lecture. So (s_n) is Cauchy, hence it converges.

(b) No. Let us produce a counterexample. Consider the sequence (s_n) whose terms are $s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Then $|s_{n+1} - s_n| = \frac{1}{n+1} < \frac{1}{n}$. However, we claim s_n does not converge (hence cannot be Cauchy). For observe that

$$
\begin{aligned}
s_1 &= 1 \\
s_2 &= 1 + \frac{1}{2} \\
s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} > 1 + 2\left(\frac{1}{2}\right) \\
s_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 1 + 3\left(\frac{1}{2}\right)
\end{aligned}
$$

And an inductive argument shows that $s_{2^n} \geq 1 + \frac{n}{2}$. So the terms s_n are arbitrarily large, and s_n diverges to ∞.

- (10.7) Let S be a bounded nonempty subset of $\mathbb{R}$ such that the supremum $s' = \sup S$ is not an element of S. For any $n \in \mathbb{N}$, the difference $s' - \frac{1}{n}$ is less than s', hence is not an upper bound for S. Therefore we can find an element $x_n \in S$ such that $x_n > s' - \frac{1}{n}$, or equivalently $\frac{1}{n} > s' - x_n$. Choose one such element for each n, and consider the sequence $(x_n) = (x_1, x_2, \cdots)$. We claim the limit of (x_n) is s'. For given $\epsilon > 0$, let $N = \frac{1}{\epsilon}$. Then for $n > N$, $|s' - x_n| = s' - x_n < \frac{1}{n} < \epsilon$. Since ϵ was arbitrary, we are done.
- (10.10) Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.

  (a) We compute $s_2 = \frac{2}{3}$, $s_3 = \frac{5}{9}$, and $s_4 = \frac{14}{27}$.

  (b) We claim that $s_n > \frac{1}{2}$ for all n. Since $s_1 = 1$, the claim is true in the base case $n = 1$. Now let us assume the claim is true for n and try to show that it holds for $n + 1$. We have $s_{n+1} = \frac{1}{3}(s_n + 1) < \frac{1}{3}(\frac{1}{2} + 1) = \frac{1}{3}(\frac{3}{2}) = \frac{1}{2}$. Ergo if the claim is true for n, it is true for $n+1$. Therefore the inductive step is true and we are done.

(c) We claim (s_n) is decreasing. It suffices to show that $s_{n+1} - s_n$ is in general a negative number. We compute $s_{n+1} - s_n = \frac{1}{3}(s_n + 1) - s_n = \frac{1}{3} - \frac{2}{3}(s_n) < \frac{1}{3} - \frac{2}{3}(\frac{1}{2}) = 0$. Here the third step follows from part (b), i.e. $s_n < \frac{1}{2}$ for all n. Ergo $s_{n+1} - s_n < 0$ for all n, so $s_{n+1} < s_n$ for all n, and s_n is decreasing.

(d) Because (s_n) is decreasing and bounded below, (s_n) must converge to some limit s. Therefore we can take the limit of both sides of the relationship $s_{n+1} = \frac{1}{3}(s_n + 1)$ using the limit laws, obtaining $s = \frac{1}{3}(s + 1)$. Therefore, $\frac{2}{3}s = \frac{1}{3}$, so $s = \frac{1}{2}$.

3. (a) By exercise (8.9) in Ross, if (s_n) is a convergent sequence of elements of $[a, b]$, then $\lim s_n$ is also in $[a, b]$. Ergo $[a, b]$ is closed.

   (b) One example of a closed unbounded subset of $\mathbb{R}$ is $\mathbb{Z}$; the only sequences in $\mathbb{Z}$ which have limits are those sequences which are eventually constant.

   (c) Let S be closed and bounded above. By exercise (10.7), there is a sequence of points (s_n) in S such that $\lim s_n = \sup S$. Therefore since S is closed, $\sup S \in S$. Therefore S has a maximum.

4. (a) Since $s_{n+1} - s_n = \frac{1}{2^{n+1}} > 0$, (s_n) is increasing. We note that $2^n s_n = 2^n + 2^{n-1} + \cdots 2 + 1 = 2^{n+1} - 1$, so in fact $s_n = 2 - \frac{1}{2^n} < 2$, and (s_n) is bounded. (Later we will introduce a general formula for this sort of computation.)

   (b) Since (s_n) is monotone bounded, it converges to some limit s, so we can take the limit of both sides of $s_{n+1} = \frac{1}{2}s_n + 1$ to obtain $s = \frac{s}{2} + 1$, implying that $s = 2$.

   (c) This sequence diverges, so the limit laws do not apply!