Homework 3 Solutions

MTH 320

- (4.7) (a) Let S, T nonempty subsets of $\mathbb{R}$ with $S \subset T$. Then we claim $\inf T \leq \inf S \leq \sup(S) \leq \sup(T)$. Proof: For any $s \in S$, since $s \in T$, $s \geq \inf T$. Therefore $\inf(T)$ is a lower bound for S, so by definition $\inf T \leq \inf S$. If T is not bounded below, $\inf T$ is $-\infty$, and therefore $\inf T \leq \inf S$ in any case. Similarly, $\sup S \leq \sup T$. So, since we know $\inf(S) \leq \sup(S)$ in general, we obtain the desired chain of inequalities.

  (b) Let $y = \max\{\sup S, \sup T\}$. Without loss of generality, we can assume this maximum is $\sup S$ (if it is not, we can switch which set is S and which is T). Suppose $a \in S \cup T$, then either $a \in S$ and $a \leq \sup S$, or $a \in T$ and $a \leq \sup T$. In either case, $a \leq y = \max\{\sup S, \sup T\}$, so y is an upper bound for $S \cup T$. Now suppose z is another lower bound for $S \cup T$, and $z < y = \sup S$. Then if $a \in S$, $a \in S \cup T$, so $a \leq z$. Therefore z is an upper bound for S that is less than $y = \sup S$. And this is nonsense. Therefore there is no such z, and y is the supremum of $S \cup T$.

- (4.11) We claim the interval (a, b) contains infinitely many rationals. Proof: By the density lemma proved in class, (a, b) contains a rational q_1. But then the interval (a, q_1) also contains a rational, call it q_2, which is different from q_1 and lies in (a, b). And the inteval (a, q_2) contains a third rational, q_3, which is different from q_1 and q_2 and lies in (a, b). Iterating this process gives an infinite set of rationals $q_1, q_2, q_3 \cdots$ in (a, b).

- (4.12) We claim the interval (a, b) contains an irrational. Proof: The interval $(a - \sqrt{2}, b - \sqrt{2})$ contains a rational, call it r. Then (a, b) contains $r + \sqrt{2}$, which must be irrational. For if $r + \sqrt{2}$ is rational, then $(r + \sqrt{2}) + (-r) = \sqrt{2}$ is the sum of two rationals and therefore rational, which is nonsense.

- (4.14) Let A and B be nonempty bounded subsets of $\mathbb{R}$, and let $A + B$ be the set of all sums $a + b$ where $a \in A$ and $\underline{i}nB$.

  (a) We want to show $\sup(A + B) = \sup(A) + \sup(B)$. By definition, if $a \in A$, then $a \leq \sup_A$, and if $b \in B$, $b \leq \sup_B$. Ergo if $a + b \in A + B$, we have $a + b \leq \sup_A + \sup_B$. Therefore $\sup_A + \sup_B$ is an upper bound for $A + B$, so $\sup_{A+B} \leq \sup_A = \sup_B$. Now, let b' be any element of B. For any $a \in A$, $a + b' \leq \sup(A + B)$, so $a \leq \sup(A + B) - b'$. Ergo since a was any element of A, $\sup(A + B) - b'$ is an upper bound for A. Therefore $\sup A \leq \sup(A + B) - b'$. Rearranging, we see that $b' \leq \sup(A + B) - \sup A$. Hence since b' was an arbitrary element of B, $\sup(A + B) - \sup A$ is an upper bound for B, implying that $\sup B \leq \sup(A + B) - \sup A$. Ergo $\sup A + \sup B \leq \sup(A + B)$. We conclude that $\sup A + \sup B = \sup(A + B)$.

- (8.1)(b). We claim that $\lim \frac{1}{n^{\frac{1}{3}}} = 0$. Proof: Given $\epsilon > 0$, let $N = \frac{1}{(\epsilon)^3}$. Then for all $n \geq N$, we have $n \geq \frac{1}{(\epsilon)^3}$, which implies $\frac{1}{n} \leq \epsilon^3$, which in turn implies that $\frac{1}{n^{\frac{1}{3}}} \leq \epsilon$. Therefore $|\frac{1}{n^{\frac{1}{3}}} - 0| = \frac{1}{n^{\frac{1}{3}}} < \epsilon$ whenever $n \geq N$, so we are done.

- 8.2 (d) We claim that the limit of the sequence $d_n = \frac{2n+4}{5n+2}$ is $\frac{2}{5}$. Proof: Given $\epsilon > 0$, let $N = \frac{1}{5}\left(\frac{16}{5\epsilon} - 2\right)$. Then for all $n \geq N$, we have $n \geq \frac{1}{5}\left(\frac{16}{5\epsilon} - 2\right)$, implying that $5n+2 \geq \frac{16}{5\epsilon}$. Therefore for each $n \geq N$, we have $\epsilon \geq \frac{16}{5(5n+2} = |\frac{16}{5(5n+2)}| = |\frac{(10n+20)-(10n+4)}{5(5n+2)}| = |\frac{2n+4}{5n+2} - \frac{2}{5}|$.

- 8.2 (e) We claim that the limit of the sequence $s_n = \frac{1}{n}\sin n$ is 0. Proof: Observe that $|s_n - 0| = |\frac{1}{n}\sin n| \leq \frac{1}{n}$. Now, given $\epsilon > 0$, choose N such that $\frac{1}{N} < \epsilon$. Then for $n > N$, $|s_n - 0| \leq \frac{1}{n} < \frac{1}{N} < \epsilon$.

- (8.3) Let s_n be a sequence of nonnegative reals with limit zero. Given $\epsilon > 0$, we can choose N such that $n > N$ implies that $|s_n - 0| < \epsilon^2$, or equivalently $s_n < \epsilon^2$. Then for $n > N$, we have $|\sqrt{s_n} - 0| = \sqrt{s_n} < \epsilon$. So $\sqrt{s_n} \to 0$.

- (8.5) (a) By assumption, $a_n \leq s_n \leq b_n$ for all n. Subtracting s gives $a_n - s \leq s_n - s \leq b_n - s$. Therefore $|s_n - s| \leq \max(|a_n - s|, |b_n - s|)$. Now since $\lim a_n = s$, there exists N_1 such that $n > N_1$ implies $|a_n - s| \leq \epsilon$, and similarly since $\lim b_n = s$, there exists N_2 such that $n > N_2$ implies $|b_n - s| \leq \epsilon$. Therefore for any $n \geq \max\{N_1, N_2\}$, we have $|s_n - s| \leq \max(|a_n - s|, |b_n - s|) < \epsilon$. Ergo $\lim s_n = s$.

  (b) Observe that $-t_n \leq s_n \leq t_n$ for all n. We know $\lim t_n = 0$; moreover, we claim that $\lim(-t_n) = 0$. For, given $\epsilon > 0$, by assumption we can choose N such that $n > N$ implies $|t_n - 0| = |t_n| < \epsilon$, so in fact $|-t_n - 0| = |t_n| < \epsilon$. So since zero is the limit of (t_n) and $(-t_n)$, by the squeeze theorem $\lim s_n = 0$ as well.

- (8.9) (a) Suppose $s_n \geq a$ for all but finitely many n. Then there is some N_1 such that $n \geq N_1$ implies that $s_n \geq a$. Now suppose, for the sake of producing a contradiction, that $\lim s_n = s$ for some number $s < a$. Then let $\epsilon = s - a$. By assumption, there exists some N_2 such that $n \geq N_2$ implies $|s_n - s| < \epsilon$. Let $n \geq \max(N_1, N_2)$. Then for $n \geq N$, we have $s_n \geq a > s$, so $(s_n - s) = (s_n - a) + (a - s) \geq a - s = \epsilon$. So in fact $|s_n - s| \geq \epsilon$, a contradiction. Therefore $\lim s_n \geq a$.

  (b) Extremely similar.

  (c)Follows immediately.

- (9.1) (b) Observe that $\frac{3n+7}{6n-5} = \frac{3+\frac{7}{n}}{6-\frac{5}{n}}$. Now, $(\frac{1}{n})$ converges to 0, so by Theorem 9.2, $\frac{7}{n} \to 0$. Similarly $\frac{5}{n} \to 0$. Now, (3) converges to 3, so by Theorem 9.3, $3 + \frac{7}{n} \to 3 + 0 = 3$. Similarly $6 - \frac{5}{n} \to 6$. Since $6 - \frac{5}{n} \neq 0$ for all n and $6 \neq 0$, Theorem 9.6 implies that $\frac{3+\frac{7}{n}}{6-\frac{5}{n}} \to \frac{3}{6} = \frac{1}{2}$.

- (9.3) Suppose $\lim a_n = a$, $\lim b_n = b$. Then since $\lim a_n$ exists, by Theorem 9.4, $\lim a_n^3 = (\lim a_n)\dot{(}\lim a_n)\dot{(}\lim a_n) = a^3$, and by Theorem 9.2, $\lim 4a_n = 4 \lim a_n = 4a$. Therefore by Theorem 9.3, $\lim(a_n^3 + 4a_n) = \lim a_n^3 + \lim 4a_n = a^3 + 4a$. Similarly, $\lim b_n^2 + 1 = b^2 + 1$, and since $b^2 \geq 0$, we know $b^2 + 1 > 0$. Ergo we may apply Theorem 9.6 to conclude that $\lim \frac{a_n^3+4a_n}{b_n^2+1} = \frac{\lim(a_n^3+4a_n)}{\lim(b_n^2+1)} = \frac{a^3+4a}{b^2+1}$.