Homework 2 Solutions

MTH 320

3. Problems from Ross.

   - (3.4)(a) Claim: In an ordered field, $0 < 1$. Proof: Observe that $1 = 1 \times 1 = (1)^2$ is always a square element of the field. By part (iv) of Theorem 3.2, all squares are greater than or equal to zero in an ordered field, so $0 \leq 1$. Moreover, we know that $0 \neq 1$ in any field (in particular, if $0 = 1$ then for any $a \in F$, $0 = a(0) = a(1) = a$ for all $a \in F$). So $0 < 1$.

     (b) Claim: If $0 < a < b$, then $0 < b^{-1} < a^{-1}$. Proof: By part (vi), we know that if $a, b > 0$ then $a^{-1}, b^{-1} > 0$, so it remains to establish the relationship of a^{-1} and b^{-1}. Suppose that $0 < a < b$, but $b^{-1} \geq a^{-1}$. Then since $b > 0$, $b(b^{-1}) \geq ba^{-1}$, so $1 > ba^{-1}$. But $b > a$, so we see that $ba^{-1} > aa^{-1} = 1$. This implies $1 > 1$, a contradiction.

   - (3.7) (a) Claim: $|b| < a$ if and only if $-a < b < a$. Proof: First, suppose $|b| < a$. Then a is positive. If $b \geq 0$, $-a < 0 < b$, whereas if $b < 0$, $-b = |b| < a$, so by Theorem 3.2(i), $b > -a$. Ergo in either case $-a < b < a$. Conversely, suppose $-a < b < a$. Then if b is positive, $|b| = b < a$, whereas if b is negative, $|b| = -b$, so since $-a < b$, by Theorem 3.2(i), $a > -b = |b|$.

     (b) Claim: $|a - b| < c$ if and only if $b - c < a < b + c$. Proof: First, observe that by the fourth axiom O4 for an ordered field $b - c < a < b + c$ is equivalent to $-c < a - b < c$. We may then apply part (a) to obtain the result.

     (c) Very similar to the above.

   - (3.8) Claim: Let $a, b \in \mathbb{R}$. Then if $a \leq b_1$ for every $b_1 > b$, then $a \leq b$. Proof: Suppose not. Then $a > b$. Let $b_1 = \frac{a+b}{2}$. Then $a > b_1 > b$, but by the hypotheses of the claim, since $b_1 > b$, $a \leq b_1$. Contradiction.

4. Problems (4.1 – 4.4) in Ross for (a), (b), (r), (m), and (w).

   Here are the supremum and infimum of each set; for each set any three numbers greater than or equal to the supremum will do for upper bounds, and similarly for the infimum and lower bounds.

   (a) The supremum of $[0, 1]$ is 1 and the infimum is 0.

(b) The supremum of $(0,1)$ is 1 and the infimum is 0.

(r) Observe that $\cap_{n=1}^{\infty}(1-\frac{1}{n}, 1+\frac{1}{n}) = 1$. Therefore the supremum and infimum of S are each 1.

(m) The supremum of $\{r \in \mathbb{Q} : r^2 < 4\}$ is 2, and the infimum is -2.

(w) Observe that $\{\sin\left(\frac{n\pi}{3}\right) : n \in \mathbb{N}\} = \{\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0\}$. Therefore this is a finite set; the supremum (and maximum) is $\frac{\sqrt{3}}{2}$ and the infimum (and minimum) is $-\frac{\sqrt{3}}{2}$.

5. (a) Suppose there are two additive identity elements in F, 0 and $0'$, such that for all $a \in F$, $a+0 = a$ and $a+0' = a$. Then setting $a = 0'$ in the first equation gives $0'+0 = 0$, but setting $a = 0'$ in the second equation gives $0+0' = 0'$. Ergo $0 = 0+0' = 0'+0 = 0'$, where we have used the commutativity of addition in a field.

   Now suppose some element a of F has two additive inverses, $-a$ and $-a'$, such that $a + -a = 0$ and $a + -a' = 0$. Then $-a = -a+0 = -a+(a+-a') = (-a+a)+-a' = 0 + -a' = -a'$, so in fact $-a = -a'$. (We could have also used Theorem 3.1(a) to cancel a from $a + -a = a + -a'$.)

   (b) Extremely similarly for multiplication.

6. (a)First, $\mathbb{C}$ is clearly closed under addition and multiplication. We check the field axioms for $\mathbb{C}$. Let $a+bi, c+di, e+fi$ be three elements of $\mathbb{C}$.

   (A1) We see that

$$\begin{aligned}(a+bi)+(c+di+e+fi) &= a+bi+[(c+e)+(d+f)i] \\ &= [a+(c+e)]+[b+(d+f)]i \\ &= [(a+c)+e]+[(b+d)+f]i \\ &= [(a+c)+(b+d)i]+(e+fi) \\ &= [(a+bi)+(c+di)]+(e+fi).\end{aligned}$$

   So addition of complex numbers is associative.

   (A2)We see that $(a+bi)+(c+di) = (a+c)+(b+d)i = (c+a)+(d+b)i = (c+di)+(a+bi)$, so addition of complex numbers is commutative since addition of real numbers is.

   (A3) The additive identity element is $0+0i$; observe that $a+bi+0+0i = (a+0)+(b+0)i = a+bi$.

   (A4) The additive inverse of any element $a+bi$ is $(-a)+(-b)i$; observe that $(a+bi)+((-a)+(-b)i) = (a+-a)+(b+-b)i = 0+0i$.

(M1) We see that

$$\begin{aligned}(a+bi)[(c+di)(e+fi)] &= (a+bi)[(ce-df)+(cf+de)i] \\ &= [a(ce-df)-b(cf+de)]+[a(cf+de)+b(ce-df)]i \\ &= (ace-adf-bcf-bde)+(acf+ade+bce-bdf)i \\ &= [(ac-bd)e-(ad+bc)f]+[(ad+bc)e+(ac-bd)f]i \\ &= [(ac-bd)+(ad+bc)i](e+fi) \\ &= [(a+bi)(c+di)](e+fi).\end{aligned}$$

Ergo multiplication of complex numbers is associative. Observe that we have used associativity, commutativity, and the distributive law for real numbers in this argument.

(M2)We see that $(a+bi)(c+di) = (ac-bd)+(ad+bc)i = (ca-db)+(da+cb)i = (c+di)(a+bi)$, so multiplication of complex numbers is commutative.

(M3) The multiplicative identity element is $(1+0i)$; observe that for any $a+bi$, we have $(a+bi)(1+0i) = (a(1)-b(0))+(a(0)+b(1))i = a+bi$.

(M4) The multiplicative inverse of any element $a+bi$ is $\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$; observe that $(a+bi)\left(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i\right) = \frac{a(a)-(b)(-b)}{a^2+b^2} + \frac{ab-ab}{a^2+b^2}i = 1+0i$.

(DL) We see that

$$\begin{aligned}(a+bi)[(c+di)+(e+fi)] &= (a+bi)[(c+e)i+(d+f)i] \\ &= [a(c+e)-b(d+f)]+[a(d+f)+b(c+e)]i \\ &= (ac+ae-bd-bf)+(ad+af+bc+be)i \\ &= [(ac-bd)+(ad+bc)i]+[(ae-bf)+(af+be)i] \\ &= (a+bi)(c+di)+(a+bi)(e+di).\end{aligned}$$

So multiplication of complex numbers distributes over addition.

(b) Suppose $\leq$ is an order relation on $\mathbb{C}$. Then by Theorem (3.2)(v), $1 > 0$. Therefore, since -1 is the additive inverse of 1, $-1 < 0$, by Theorem 3.2(i). However, by Theorem 3.2 (v), all squares in an ordered field are nonnegative, so $i^2 = -1 \geq 0$. This is a contradiction, as an element cannot be both less than zero and greater than or equal to zero.

7. Say that $A = \{a_1, a_2, a_3, \cdots\}$ and $B = \{b_1, b_2, b_3, \cdots\}$. Then we can mimic the proof in class that $\mathbb{N}\times\mathbb{N}$ is countable, using diagonals in the first quadrant of the plane to build a list of elements of $A\times B$. Specifically, this list goes $\{(a_1,b_1),(a_1,b_2),(a_2,b_1),(a_1,b_3),(a_2,b_2),(a_3,b_1),\cdots$ In more formal language, we already know there is a bijection between $\mathbb{N}\times\mathbb{N}$ and $\mathbb{N}$, so we can use the existing bijections $A\to\mathbb{N}$ and $B\to\mathbb{N}$ to get a bijection $A\times B\to\mathbb{N}\times\mathbb{N}\to\mathbb{N}$.

8. Suppose for the sake of producing a contradiction that we can make a list $\{A_1, A_2, \cdots\}$ of the subsets of $\mathbb{N}$. We can associate to any A_i of a binary sequence by letting $a_{ij} = 1$ if j is in A and $a_{ij} = 0$ if j is not in A. But then if we set b_i to be 0 if $a_{ii} = 1$ and vice versa, the subset $B \subset \mathbb{N}$ determined by $b_1, b_2, \cdots$ does not appear in our list of subsets A_i. Ergo there are uncountably many subsets of $\mathbb{N}$.