Homework 1 Solutions

MTH 320

3. Problems from Ross.

- (1.4) (a) Let $s_n = 1+3+\cdots+(2n-1)$ for all $n \in \mathbb{N}$. We see that for $n = 1, 2, 3, 4$ we have $s_n = 1, 4, 9, 16$ respectively. We therefore guess that $s_n = n^2$ in general.

  (b) We proceed by induction. The base case $s_1 = 1 = 1^2$ is true. Now let us suppose that we already know $s_n = n^2$, and attempt to prove that $s_{n+1} = (n+1)^2$. We see that

  $$\begin{aligned} s_{n+1} &= 1 + 3 + \cdots + (2(n+1) - 1) \\ &= s_n + (2n+1) \\ &= n^2 + (2n+1) \\ &= n^2 + n + (n+1) \\ &= n(n+1) + (n+1) \\ &= (n+1)(n+1) \\ &= (n+1)^2. \end{aligned}$$

  Thus the inductive step is also proven, and the claim is true.

- (1.8)(a) We proceed by induction. Here the statement P_n is $n^2 > n+1$. Our base case is $n = 2$, so we check that P_2 is true: $2^2 = 4 > 3 = 2+1$. We now proceed to the inductive step. Suppose that P_n is true, for some $n \geq 2$. That is, we assume $n^2 > n+1$. Then we have

  $$\begin{aligned} (n+1)^2 &= n(n+1) + (n+1) \\ &= n^2 + n + (n+1) \\ &> (n+1) + n + (n+1) \\ &= 3n + 2 \\ &> n + 2. \end{aligned}$$

  Ergo if P_n is true, P_{n+1} is true, so we are done.

  (b) We proceed by induction. Here the statement P_n is $n! > n^2$. Our base case is $n = 4$, so we check P_4 is true: $4! = 24 > 16 = 4^2$. Now suppose P_n is true. Then we have

  $$\begin{aligned} (n+1)! &= (n+1) \times n \times \cdots \times 1 \\ &= (n+1)n! \\ &> (n+1)n^2 \\ &> (n+1)(n+1) \\ &= (n+1)^2. \end{aligned}$$

Here we have used part (a), which tells us that for $n \geq 2$, $n^2 > n+1$. Ergo for $n \geq 4$, if P_n is true, then P_{n+1} is true. Thus the inductive step is proved, and the claim is true.

- (1.11) (a) Suppose P_n is true. Then $n^2 + 5n + 1$ is an even integer. To show that P_{n+1} is true, we consider the integer $(n+1)^2 + 5(n+1) + 1$. We see that

$$\begin{aligned}(n+1)^2 + 5(n+1) + 1 &= n(n+1) + (n+1) + 5n + 5 + 1 \\ &= n^2 + n + n + 1 + 5n + 6 \\ &= (n^2 + 5n + 1) + 2n + 6.\end{aligned}$$

  Since $2n + 6$ is an even number, and $n^2 + 5n + 1$ is even by assumption, we see that if P_n is true, then P_{n+1} is also true.

  (b) No n whatsoever. Observe that $n^2 + 5n + 1 = n(n+5) + 1$. Exactly one of n and $n+5$ is even, so the product $n(n+5)$ is even, and therefore $n(n+5)+1$ is odd. The moral is that it is very important to check that a base case is true when using induction.

4. We (again) proceed by induction. The base case P_2 is $n = 2$, which is true since $(1+x)^2 = 1 + 2x + x^2 > 1 + 2x$. Now assume that the nth case P_n is true, and consider $(1+x)^{n+1} = (1+x)(1+x)^n$. Because $1 + x > 0$ and $(1+x)^n > 1 + nx$ by assumption, we see that $(1+x)^{n+1} > (1+x)(1+nx) = 1 + (n+1)x + nx^2 > 1 + (n+1)x$, where the last step follows because nx^2 is a positive number. Therefore if P_n is true, P_{n+1} is also true, and the claim follows.

5. (2.2) We see that $2^{\frac{1}{3}}$ is a root of $f(x) = x^3 - 2 = 0$. According to the Rational Zeroes Theorem, the only rational numbers which are solutions of $f(x) = 0$ are of the form $\frac{c}{d}$ where c divides the constant term, -2 and d divides the leading coefficient, 1. Therefore the only possible rational zeroes of $f(x)$ are ± 1 and ± 2, all of which are clearly not $2^{\frac{1}{3}}$. We conclude that $2^{\frac{1}{3}}$ cannot be a rational number. The other cases proceed extremely similarly.

6. (2.3) We would like to find an polynomial with integer coefficients which has $a = \sqrt{2 + \sqrt{2}}$ as a zero. We have $a^2 = 2 + \sqrt{2}$; working backward from the quadratic formula, we see that a^2 is a zero of $g(x) = x^2 - 4x + 2$. Hence a is a zero of $f(x) = g(x^2) = x^4 - 4x^2 + 2$. According to the Rational Zeroes Theorem, the only rational numbers which could be solutions of $f(x) = 0$ are of the form $\frac{c}{d}$ where c divides the constant term, 2 and d divides the leading coefficient, 1. Therefore the only possible rational zeroes of $f(x)$ are ± 1 and ± 2, all of which are clearly not a (and

indeed, not actually zeroes of $f(x)$). We conclude that a cannot be a rational number.

7. (a)The inductive step is flawed when $n = 2$. In particular, suppose we have a set of two horses $\{x_1, x_2\}$. Then the set $A_1 = \{x_1\}$ and $A_2 = \{x_2\}$ have no overlap. Therefore we cannot conclude that any two horses have the same color, and thus cannot induct to larger sets of horses (even though the inductive step is valid for higher n).

   (b) One correct answer is that this is not a good proof because we have failed to give a definition of "interesting," and indeed, seem to have changed whatever definition we were using midway through the argument. This illustrates the importance of defining mathematical terms carefully instead of relying upon their colloquial meanings.