Homework 13 Solutions

MTH 320

1. Problems from Ross.

(30.1)(a) We observe that $e^{2x} - \cos x \to 1 - 1 = 0$ as $x \to 0$ and $x \to 0$ as $x \to 0$. Ergo we use L'Hospital's Rule.

$$\lim_{x\to 0} \frac{e^{2x} - \cos x}{x} = \lim_{x\to 0} \frac{2e^{2x} + \sin x}{1} = 2$$

(30.2)(c) We compute

$$\lim_{x\to 0} \frac{1}{\sin x} - \frac{1}{x} = \lim_{x\to 0} \frac{x - \sin x}{x \sin x} = \lim_{x\to 0} \frac{1 - \cos x}{\sin x + x\cos x} = \lim_{x\to 0} \frac{\sin x}{2\cos x - x \sin x} = \frac{0}{2} = 0.$$

where the second and third equalities are applications of L'Hospital's Rule, both using the case where the numerator and denominator both approach zero.

(30.3)(c) We see that as $x \to 0^+$, we have $1 + \cos x \to 2$ and $e^x - 1 \to 0$, and the quotient $\frac{1+\cos x}{e^x - 1}$ is positive. So $\lim_{x\to 0^+} \frac{1+\cos x}{e^x-1} = \infty$. (Note that using L'Hospital's Rule on this problem returns a false answer, since the hypotheses are not satisfied.)

(30.5)(a) We rewrite $(1+2x)^{\frac{1}{x}}$ as $e^{\frac{\ln(1+2x)}{x}}$. Then using L'Hospital's Rule we have

$$\lim_{x\to 0} \frac{\ln(1+2x)}{x} = \lim_{x\to 0} \frac{2}{1+2x} = 2.$$

Ergo

$$\lim_{x\to 0} (1+2x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{\ln(1+2x)}{x}} = e^2.$$

(26.2) (a) We have $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ on $(-1,1)$, so differentiating both sides gives $\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$. Multiplying by x we see that $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$.

(b) We observe that $\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \frac{(1/2)}{(1-(1/2))^2} = 2$.

(c) Similarly to part (b) we have $\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{3}{4}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n} = -\frac{3}{16}$.

(26.4) (a) Yes, since $e^x = \sum_{n=1}^{\infty} \frac{1}{n!} x^n$, we have $e^{-x^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} x^{2n}$.

(b) We integrate the power series to obtain $\int_0^x e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n 2n}{n!} x^{2n-1}$. Note that the reason this is interesting is that e^{x^2} does not have an elementary antiderivative.

(26.7) No. We would expect any function $f(x) = \sum a_n x^n$ to be infinitely differentiable, and $f(x) = |x|$ fails to be differentiable at $x = 0$.

(31.1) The derivatives of $f(x) = \cos x$ are $f^(0)(x) = \cos x$, $f^(1)(x) = -\sin x$, $f^(2)(x) = -\cos x$, $f^(3)(x) = \sin x$, and subsequently $f^(4j+k)(x) = f^(k)(x)$. Therefore the derivatives of $\cos x$ at 0 cycle through $1, 0, -1, 0, 1, 0, -1, 0, \cdots$. Therefore the Taylor expansion of $\cos x$ about 0 is given by

$$T(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Moreover, all derivatives of $f(x) = \cos x$ on $\mathbb{R}$ are bounded; that is, $|f^(n)(x)| \leq 1$ for all $x \in \mathbb{R}$, so by Taylor's Theorem the power series above converges to $\cos x$.

2. *The five constants.*

(a) We see $i^3 = -i$, $i^4 = 1$, and in general $i^{4k+j} = i^j$, that is, the powers of i cycle through $i, -1, -i, 1, i, -1, -i, 1, \cdots$.

(b)We compute that

$$\begin{aligned}
e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \\
&= 1 + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \cdots \\
&= (1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!} \\
&= \cos x + i \sin x
\end{aligned}$$

(c) Letting $x = \pi$, we obtain $e^{i\pi} = -1 + i(0)$, or $e^{i\pi} + 1 = 0$ as desired.